EE 464

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Lecture Notes Part 10a

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10.0 Two Random Variables

Given two random variables, X and Y, we can regard each pair of outcomes as an element in 2-dimensional space.

10.1 Joint Distribution and Density

Definition: The *joint distribution function*, $F_{XY}(x,y)$, of two random variables, X, Y, is defined by

$$F_{XY}(x,y) = P(X \le x, Y \le y).$$

We can write $F(x,y)$ for $F_{XY}(x,y)$ when clear to do so.

Plots of events will be given in class.

Properties:

i. $P(X \le x, Y \le y) \le P(X \le x), \quad P(X \le x, Y \le y) \le P(Y \le y).$

ii. $P(X = -\infty, Y \le y) \le P(X = -\infty) = 0 \Rightarrow F(-\infty, y) = 0.$

iii. Similarly, $F(x, -\infty) = 0.$

iv. $P(X \le \infty, Y \le \infty) = P(\Omega) = 1 \Rightarrow F(\infty, \infty) = 1.$

v. $\{x_1 < X \le x_2, Y \le y\} = \{X \le x_2, Y \le y\} \setminus \{X \le x_1, Y \le y\}$ which implies

$$\begin{aligned} P\left(x_1 < X \le x_2, Y \le y\right) &= P\left(X \le x_2, Y \le y\right) - P\left(X \le x_1, Y \le y\right) \\ &= F(x_2, y) - F(x_1, y). \end{aligned}$$

vi. Similarly,

$$\begin{aligned} P\left(X \le x, y_1 < Y \le y_2\right) &= P\left(X \le x, Y \le y_2\right) - P\left(X \le x, Y \le y_1\right) \\ &= F(x, y_2) - F(x, y_1). \end{aligned}$$

vii. $\{x_1 < X \le x_2,\ y_1 < Y \le y_2\}$
$= \{x_1 < X \le x_2, Y \le y_2\} \setminus \{x_1 < X \le x_2, Y \le y_1\}$
which combined with the above yields

$$P(x_1 < X \le x_2,\ y_1 < Y \le y_2) = F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1).$$

Definition: The *joint density function*, $f_{XY}(x,y)$, of two random variables, X, Y, is defined by

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}.$$

We can write $f(x,y)$ for $f_{XY}(x,y)$ when clear to do so.

Therefore,

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(u,v) du dv.$$

Recall,

$$F(\infty,\infty) = 1 \Rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(u,v) du dv = 1.$$

Definition: In the discrete case we have the *joint probabilities*

$$p_{ik} = P(X = x_i, Y = y_k).$$

Here

$$\sum_{i,k} p_{ik} = 1.$$

Definitions: For two random variables X and Y, $F_X(x)$ is called the *marginal distribution* of X and $f_x(x)$ is called the *marginal density* of X. Similarly, for $F_Y(y)$ and $f_Y(y)$.

Claim:

$$F_X(x) = F(x,\infty), \;\; F_Y(y) = F(\infty, y),$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y) dy, \;\; f_Y(y) = \int_{-\infty}^{\infty} f(x,y) dx.$$

Proof:

$$F_X(x) = P(X \le x) = P(X \le x, Y \le \infty) = F(x,\infty).$$

$$F_Y(y) = P(Y \le y) = P(X \le \infty, Y \le y) = F(\infty, y).$$

Now

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v) du dv.$$

So,

$$\frac{\partial F(x,y)}{\partial x} = \int_{-\infty}^{y} f(x,v) dv, \;\; \frac{\partial F(x,y)}{\partial y} = \int_{-\infty}^{x} f(u,y) du.$$

But,

$$F_X(x) = F(x, \infty) \text{ and } f_X(x) = \frac{d}{dx} F_X(x) = \frac{\partial}{\partial x} F(x, \infty)$$

so

$$f_X(x) = \left. \frac{\partial F(x,y)}{\partial x} \right|_{y=\infty} = \int_{-\infty}^{\infty} f(x,v) dv = \int_{-\infty}^{\infty} f(x,y) dy.$$

Similarly,

$$f_Y(y) = \left. \frac{\partial F(x,y)}{\partial y} \right|_{x=\infty} = \int_{-\infty}^{\infty} f(x,y) dx.$$

Definition: In the discrete case we have *marginal probabilities*, p_i and q_k, where $p_i = P(X = x_i)$ and $q_k = P(Y = y_k)$.

Here

$$p_i = \sum_k p_{ik}, \quad q_k = \sum_i p_{ik}.$$

Example: Suppose the two-dimensional continuous random variable (X, Y) has joint *pdf*

$$f(x,y) = \begin{cases} x^2 + \frac{xy}{3}, & 0 \le x \le 1, \ 0 \le y \le 2 \\ 0, & \text{elsewhere.} \end{cases}$$

a. Verify this is a valid *pdf*.

Obviously, $f(x,y) \ge 0$. Check

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dx dy = 1.$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(x^2 + \frac{xy}{3} \right) dx dy$$

$$= \int_0^2 \left(\frac{x^3}{3} + \frac{x^2 y}{6} \right) \Bigg|_{x=0}^{x=1} dy = \int_0^2 \left(\frac{1}{3} + \frac{y}{6} \right) dy = \left(\frac{y}{3} + \frac{y^2}{12} \right) \Bigg|_0^2 = 1.$$

b. Let $B = \{X + Y \ge 1\}$. Find $P(B)$.

$$P(B) = 1 - P(\bar{B}), \text{ where } \bar{B} = \{X + Y < 1\}.$$

Note $x + y < 1 \Rightarrow y < 1 - x$. So

$$P(B) = 1 - \int_0^1 \int_0^{1-x} \left(x^2 + \frac{xy}{3} \right) dy dx = \frac{65}{72}.$$

c. Find $f_X(x)$ and $f_Y(y)$.

$$f_X(x) = \int_{-\infty}^{\infty} \left(x^2 + \frac{xy}{3}\right) dy = \int_0^2 \left(x^2 + \frac{xy}{3}\right) dy.$$

Thus,

$$f_X(x) = \begin{cases} 2x^2 + \dfrac{2x}{3}, & 0 \le x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Also,

$$f_Y(y) = \int_0^1 \left(x^2 + \frac{xy}{3}\right) dx.$$

We get

$$f_Y(y) = \begin{cases} \dfrac{1}{3} + \dfrac{y}{6}, & 0 \le x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Definition: We say (X, Y) is *uniformly distributed* over a region R in the Euclideam plane if

$$f(x, y) = \begin{cases} \dfrac{1}{\text{area of R}}, & \text{for } (x, y) \in R, \\ 0, & \text{elsewhere.} \end{cases}$$

Example: Suppose (X, Y) is uniformly distributed over the region between the curves $y = x$ and $y = x^2$.

a. Find $f(x, y)$.

$$f(x, y) = \frac{1}{\text{area of R}},$$

where

$$\text{area of R} = \int_0^1 \int_{x^2}^{x} dy dx = \frac{1}{6}.$$

So

$$f(x, y) = \begin{cases} 6, & \text{for } (x, y) \in R, \\ 0, & \text{elsewhere.} \end{cases}$$

b. Find the marginal *pdf's* of X and Y.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{x^2}^{x} 6dy.$$

Thus,

$$f_X(x) = \begin{cases} 6(x - x^2), & 0 \le x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{y}^{\sqrt{y}} 6dx.$$

Thus,

$$f_Y(y) = \begin{cases} 6(\sqrt{y} - y), & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$