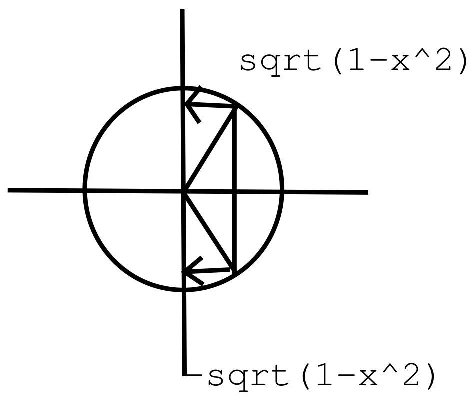
EE464: HOMEWORK 7 SOLUTIONS

Problem 1 Solution

$$\begin{aligned} X &= \cos\theta \\ Y &= \sin\theta \end{aligned}$$



$$\begin{aligned} X^2 + Y^2 &= 1 \\ Y &= \pm\sqrt{1 - X^2} \\ f_Y(y|x) &= \frac{1}{2}\delta(y + \sqrt{1 - x^2}) + \frac{1}{2}\delta(y - \sqrt{1 - x^2}) \\ E[Y|x] &= -\frac{1}{2}\sqrt{1 - x^2} + \frac{1}{2}\sqrt{1 - x^2} \\ &= 0 \ \forall x \end{aligned}$$

Problem 2 Solution

a) $E[(X + Y)^2] = E[X^2 + 2XY + Y^2] = E[X^2] + 2E[XY] + E[Y^2]$

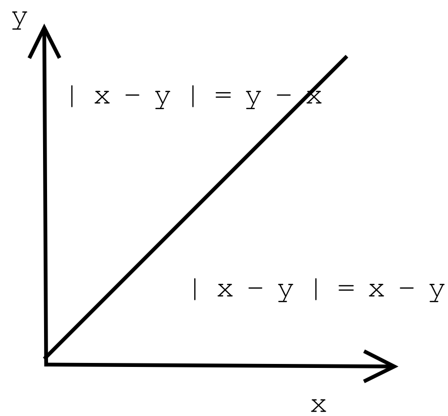
b)

$$\begin{aligned} VAR[X + Y] &= E[(X + Y)^2] - E[X + Y]^2 \\ &= E[X^2] + 2E[XY] + E[Y^2] - E[X]^2 - 2E[X]E[Y] - E[Y]^2 \\ &= VAR[X] + VAR[Y] + 2(E[XY] - E[X]E[Y]) \end{aligned}$$

c) $VAR[X + Y] = VAR[X] + VAR[Y]$ iff $E[XY] = E[X]E[Y]$, that is, if X and Y are uncorrelated.

Problem 3 Solution

For taking expectation, we need to integrate over the entire first quadrant. It can be divided in two halfs as shown in the figure below. Since the density is symmetric about x and y, we can integrate over only one half and multiply it be two.



$$
\begin{aligned}
E[|X-Y|] &= \int_0^\infty \int_0^\infty |x-y| e^{-(x+y)} dxdy \\
&= 2\int_{x=0}^{x=\infty} \int_{y=0}^{y=x} (x-y)e^{-x}e^{-y} dydx \\
&= 2\int_0^\infty e^{-x}[x(1-e^{-x}) - \int_0^x ye^{-y}dy]dx \\
&= 2\int_0^\infty [(xe^{-x} - xe^{-2x}) - e^{-x}(1-(1+x)e^{-x})]dx \\
&= 2\int_0^\infty (xe^{-x} + e^{-2x} - e^{-x})dx \\
&= 2[1 + \frac{1}{2} - 1] \\
&= 1
\end{aligned}
$$

Problem 4 Solution

The relevant parameters are $n = 1000, m = np = 500, \sigma^2 = npq = 250$. The Central Limit Theorem then gives:

$$
\begin{aligned}
P[400 \le N \le 600] &= P\left[\frac{400-500}{\sqrt{250}} \le \frac{N-m}{\sigma} \le \frac{600-500}{\sqrt{250}}\right] \\
&\approx Q(-6.324) - Q(6.324) = 1 - 2Q(6.324) \\
&= 1 - 2.54(10^{-10}) \\
P[500 \le N \le 550] &= Q(0) - Q(3.162) = \frac{1}{2} - 7.3(10^{-4})
\end{aligned}
$$

Problem 5 Solution

a)

$$\begin{aligned} E[\bar{X}] &= E(\frac{1}{n}\sum_{i=1}^{n} X_i) \\ &= \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{1}{n}n\mu \\ &= \mu \end{aligned}$$

b) First note that $VAR[aX] = E[(aX)^2] - E[aX]^2 = a^2 VAR[X]$.

$$\begin{aligned} VAR[\bar{X}] &= VAR(\frac{1}{n}\sum_{i=1}^{n} X_i) \\ &= \frac{1}{n^2}VAR(\sum_{i=1}^{n} X_i) \\ &= \frac{1}{n^2}\sum_{i=1}^{n} VAR[X_i] = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 \\ &= \frac{1}{n^2}n\sigma^2 = \frac{\sigma^2}{n} \end{aligned}$$

This is true since X_is are uncorrelated.

c) Sample variance is given by $\bar{V} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

$$\begin{aligned} E[\bar{V}] &= \frac{1}{n-1}E(\sum_{i=1}^{n}(X_i - \bar{X})^2) \\ &= \frac{1}{n-1}E(\sum_{i=1}^{n}(X_i^2 - 2X_i\bar{X} + \bar{X}^2)) \\ &= \frac{1}{n-1}E(\sum_{i=1}^{n} X_i^2 - \frac{2}{n}\sum_{j=1}^{n} X_j \sum_{i=1}^{n} X_i + \sum_{i=1}^{n}\bar{X}^2) \\ &= \frac{1}{n-1}(\sum_{i=1}^{n} E(X_i^2) - \frac{2}{n}E((\sum_{i=1}^{n} X_i)^2) + nE(\bar{X}^2) \end{aligned}$$

Now, since $E[(\sum_{i=1}^{n} X_i)^2] = n^2 E[(\frac{1}{n}\sum_{i=1}^{n} X_i)^2] = n^2 E[\bar{X}^2]$ $= n^2(VAR(\bar{X}) - E[\bar{X}]^2) = n\sigma^2 - n^2\mu^2$. Substituting this is above equation and using part a and part b,

$$\begin{aligned} E[\bar{V}] &= \frac{1}{n-1}(\sum_{i=1}^{n} E(X_i^2) - \frac{2}{n}E((\sum_{i=1}^{n} X_i)^2) + nE(\bar{X}^2)) \\ &= \frac{1}{n-1}(n(\sigma^2 + \mu^2) - \frac{2}{n}(n\sigma^2 - n^2\mu^2) + n(\frac{\sigma^2}{n} + \mu^2) \\ &= \frac{1}{n-1}((n - 2 + 1)\sigma^2 + (n - 2n + n)\mu^2) \\ &= \sigma^2 \end{aligned}$$

Problem 6 Solution
Note that $Z = \frac{X-2}{3}\ N(0,1)$
a)

$$\begin{aligned} P[X < 5] &= P[\frac{X-2}{3} < \frac{5-2}{3}] \\ &= P[Z < 1] \\ &= \frac{1}{2} + erf(1) \\ &= 0.5 + 0.34134 = 0.84134 \end{aligned}$$

b)

$$\begin{aligned} P[X > -1] &= 1 - P[X \leq -1] \\ &= 1 - P[\frac{X-2}{3} \leq \frac{-1-2}{3}] \\ &= 1 - P[Z \leq -1] \\ &= P[Z < 1] = \frac{1}{2} + erf(1) \\ &= 0.84134 \end{aligned}$$

(Draw figure to observe that $1 - P(Z \leq -1) = P(Z > -1) = P(Z < 1))$ Note that $Z\ N(0,1)$ is symmetric.)
c)

$$\begin{aligned} P[-1 < X < 5] &= P[\frac{-1-2}{3} < \frac{X-2}{3} < \frac{5-2}{3}] \\ &= P[-1 < Z < 1] \\ &= P[Z < 1] - P[Z \leq -1] \\ &= P[Z < 1] - (1 - P[Z < 1]) \\ &= 2P[Z < 1] - 1 \\ &= 2(\frac{1}{2} + erf(1)) - 1 = 2erf(1) \\ &= 0.68268 \end{aligned}$$

d)

$$\begin{aligned} P[X < 10] &= P[\frac{X-2}{3} < \frac{10-2}{3}] \\ &= P[Z < 8/3] \\ &= \frac{1}{2} + erf(\frac{8}{3}) = 0.5 + 0.49608 \end{aligned}$$

Problem 7 Solution
Given $X = x$, $Y = WZ = xZ$
a) Best MSE predictor is given by $E[Y|X = x]$

$$f_{Y|X=x}(y|x) = \frac{1}{|x|} f_Z(\frac{y}{x}) = \frac{1}{x} \quad x < y < 2x$$

Therefore,

$$\begin{aligned} E[Y|X=x] &= \int_{y=x}^{y=2x} \frac{1}{x} y dy \\ &= \frac{1}{x}[\frac{4x^2}{2} - \frac{x^2}{2}] \\ &= \frac{3x}{2} \end{aligned}$$

b) Since the best MSE predictor for Y based on X is already linear, the best *linear* MSE predictor is also the same.

Problem 8 Solution
a)
Lets find the marginal density $f_Y(y)$

$$\begin{aligned} f_Y(y) &= \int_{x=0}^{x=1} (x^2 + \frac{xy}{3}) dx \\ &= [\frac{x^3}{3} + \frac{x^2 y}{6}]_{x=0}^{x=1} \\ &= \frac{1}{3} + \frac{y}{6} \quad 0 \le y \le 2 \end{aligned}$$

Now,

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} \\ &= \frac{x^2 + \frac{xy}{3}}{\frac{1}{3} + \frac{y}{6}} \\ &= \frac{6x^2 + 2xy}{2+y} \quad 0 \le x \le 1, 0 \le y \le 2 \end{aligned}$$

b)

$$\begin{aligned} E[X|Y=y] &= \int_{x=0}^{x=1} x \frac{6x^2 + 2xy}{2+y} dx \\ &= \frac{1}{2+y}[\frac{6x^4}{4} + \frac{2x^3 y}{3}]_0^1 \\ &= \frac{9+4y}{6(2+y)} \end{aligned}$$

Problem 9 Solution
Given $X = x$, $Y = Z_1 + Z_1 Z_2 = Z_1(1+Z_2) = Z_1(1+x)$
So,

$$\begin{aligned} f_{Y|X=x}(y|x) &= \frac{1}{|1+x|} f_Z(\frac{y}{1+x}) \\ &= \frac{\lambda}{1+x} e^{-\lambda y/(1+x))} U(y) \end{aligned}$$

a)

$$E[Y|X=x] = \int_0^\infty y f_{Y|X=x}(y|x)dy = \int_0^\infty y \frac{1}{1+x} e^{-\frac{\lambda y}{1+x}} dy = \frac{1+x}{\lambda}$$

b)

$$E[E[Y|X]] = E[\frac{1+X}{\lambda}] = \frac{1+E[X]}{\lambda} = \frac{1}{\lambda} + \frac{1}{\lambda^2}$$

c)

$$Var[E[Y|X]] = VAR[\frac{1+X}{\lambda}] = \frac{1}{\lambda^2} VAR[1+X] = \frac{1}{\lambda^2} VAR[X] = \frac{1}{\lambda^4}$$

d)

$$VAR[Y|X=x] = \int_{-\infty}^{\infty} y^2 f_{Y|X=x}(y|x)dy - (E[Y|X])^2 = \frac{(1+x)^2}{\lambda^2}$$

e)

$$E[VAR(Y|X)] = E[\frac{(1+X)^2}{\lambda^2}] = \frac{1}{\lambda^2} E[(1+X)^2] = \frac{1}{\lambda^2}(1 + \frac{2}{\lambda} + \frac{2}{\lambda^2})$$

f) The best MSE predictor of Y given X=x is

$$E\{Y|X=x\} = \frac{1+x}{\lambda}$$

g) Since the best MSE predictor is linear, the best *linear* MSE predictor is the same as the best MSE predictor $\frac{1+x}{\lambda}$

Best of Luck for the exams!

TA: Anand Joshi
email: ajoshi@usc.edu

Problem 9. Let Z_1 and Z_2 be independent random variables each having an exponential density of the form $f_Z(z) = \lambda e^{-\lambda z} U(z)$. Define $X = Z_2,\ Y = Z_2(1 + Z_1)$. Find

a. Find $E(Y|X = x)$.

Solution.

$$E(Y|X = x) = E(Z_2(1 + Z_1)|Z_2 = x) = E(x(1 + Z_1)) = E(x + xZ_1)$$
$$= x + xE(Z_1) = x + \frac{x}{\lambda} = x\left(1 + \frac{1}{\lambda}\right).$$

b. Find $E\left(E(Y|X)\right)$.

Solution.

$$E\left(E(Y|X)\right) = E\left[X\left(1 + \frac{1}{\lambda}\right)\right] = \frac{1}{\lambda}\left(1 + \frac{1}{\lambda}\right).$$

c. Find $Var\left(E(Y|X)\right)$.

Solution.

$$Var\left(E(Y|X)\right) = Var\left[X\left(1 + \frac{1}{\lambda}\right)\right] = \frac{1}{\lambda^2}\left(1 + \frac{1}{\lambda}\right)^2.$$

d. Find $Var\left(Y|X = x\right)$.

Solution.

$$Var\left(Y|X = x\right) = Var\left(Z_2(1 + Z_1)|Z_2 = x\right) = Var(x(1 + Z_1)) = \frac{x^2}{\lambda^2}.$$

e. Find $E\left(Var(Y|X)\right)$.

Solution.

$$E\left(Var(Y|X)\right) = E\left[\frac{X^2}{\lambda^2}\right] = \left[Var(X) + [E(X)]^2\right]\left[\frac{1}{\lambda^2}\right]$$
$$= \left[Var(Z_2) + [E(Z_2)]^2\right]\left[\frac{1}{\lambda^2}\right] = \left[\frac{1}{\lambda^2} + \frac{1}{\lambda^2}\right]\left[\frac{1}{\lambda^2}\right] = \frac{2}{\lambda^4}.$$

f. Find the best MSE predictor of Y given $X = x$.

Solution.

$$E(Y|X = x) = x\left(1 + \frac{1}{\lambda}\right).$$

g. Find the best linear MSE predictor of Y based on X.

Solution. From part (f) we see the best MSE predictor is itself linear so

$$\hat{Y} = \left(1 + \frac{1}{\lambda}\right) X.$$