EE464: HOMEWORK 6 SOLUTIONS

Problem 1 Solution

a) For $x > 0, y > 0$

$$
\begin{aligned}
F_{XY}(x,y) &= \int_0^x \int_0^y ax' e^{-ax'^2/2} by' e^{-by'^2/2} dx' dy' \\
&= (1 - e^{-ax^2/2})(1 - e^{-ay^2/2})
\end{aligned}
$$

b)

$$
\begin{aligned}
P[X > Y] &= \int_0^\infty \int_0^x axe^{-ax^2/2} bye^{-by^2/2} dydx \\
&= \int_0^\infty axe^{-ax^2/2}(1 - e^{-ax^2/2})dx \\
&= 1 - \frac{a}{a+b}
\end{aligned}
$$

c)

$$
\begin{aligned}
F_X(x) &= \lim_{y \to \infty} F_X(x,y) = 1 - e^{-ax^2/2} \quad x > 0 \\
&\Rightarrow \ f_X(x) = \frac{d}{dx} F_X(x) = axe^{-ax^2/2} \quad x > 0
\end{aligned}
$$

Similarly,

$$
f_Y(y) = bye^{-by^2/2}
$$

Problem 2 Solution

a) The probability is obtained by integrating the joint pdf over the region over which the condition is satisfied.

$$
\begin{aligned}
P[X + Y \leq 8] &= \int_0^8 \int_0^{8-x} 2e^{-x} e^{-2y} dydx \\
&= \int_0^8 e^{-x}(1 - e^{-2(8-x)})dx \\
&= 1 - 2e^{-8} + e^{-16}
\end{aligned}
$$

b)

$$\begin{aligned} P[X<Y] &= \int_0^\infty \int_x^\infty 2e^{-x}e^{-2y}dydx \\ &= \int_0^\infty e^{-x}e^{-2x}dx = \frac{1}{3} \end{aligned}$$

c)

$$\begin{aligned} P[X-Y\leq 10] &= \int_0^\infty \int_0^{y+10} e^{-x}dx2e^{-2y}dy \\ &= \int_0^\infty (1-e^{-(y+10)})e^{-2y}dy \\ &= 1-\frac{2}{3}e^{-10} \end{aligned}$$

d)

$$\begin{aligned} P[X^2<Y] &= \int_0^\infty \int_{x^2}^\infty e^{-x}2e^{-2y}dydx \\ &= \int_0^\infty e^{-x}e^{-2x^2}dx \\ &= e^{\frac{1}{8}}\int_0^\infty e^{-2(x^2+\frac{1}{2}x+\frac{1}{16})}dx \\ &= e^{\frac{1}{8}}\int_0^\infty e^{-2(x^2+\frac{1}{2}x+\frac{1}{16})}dx \\ &= e^{\frac{1}{8}}\sqrt{2\pi(\frac{1}{4})}\int_0^\infty \frac{e^{-(x+\frac{1}{4})^2/2(\frac{1}{4})}}{\sqrt{2\pi(\frac{1}{4})}} \\ &= e^{\frac{1}{8}}\sqrt{\frac{\pi}{2}}Q(\frac{1}{2}) \end{aligned}$$

Problem 3 Solution

$$\begin{aligned} f_X(x) &= \int_0^\infty xe^{-x}e^{-xy}dy = xe^{-x}(-\frac{1}{x}e^{-xy})_0^\infty = e^{-x} \\ f_Y(y) &= \int_0^\infty xe^{-x(1+y)}dx = \frac{e^{-x(1+y)}((1+y)x-1)}{(1+y)^2}\,|_0^\infty \\ &= \frac{1}{(1+y)^2} \end{aligned}$$

The limiting value at infinity can be found by using L'Hospitals rule.

Problem 4 Solution

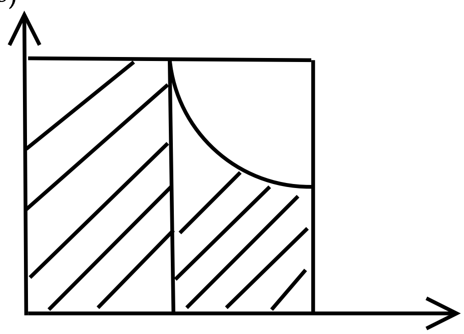
a)

$$\begin{aligned} P[X^2 < \frac{1}{2}, |Y-1| < \frac{1}{2}] &= P[X^2 < \frac{1}{2}]P[|Y-1| < \frac{1}{2}] \\ &= P[X < \frac{1}{\sqrt{2}}]P[Y > \frac{1}{2}] = \frac{1}{2}\frac{1}{\sqrt{2}} \end{aligned}$$

b)

$$P[X/2 < 1, Y > 0] = P[X < 2]P[Y > 0] = 1$$

c)



The region of integraation is shown in the figure above

$$\begin{aligned} f(x,y) &= f(x)f(y) = 1 \\ P[XY < 1/2] &= \frac{1}{2} + \int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2x}} 1.dydx \\ &= \frac{1}{2} + \int_{\frac{1}{2}}^{1} \frac{1}{2x}dx \\ &= \frac{1}{2} + \frac{1}{2}\ln(x)|_{1/2}^{1} \\ &= 0.85 \end{aligned}$$

d)

$$P[min(X,Y) > 1/3] \quad = P(X > 1/3)P(Y > 1/3) \quad = (\frac{2}{3})^2 \frac{4}{9}$$

Problem 5 Solution

$$\begin{aligned} P[X=1] &= P[X=1,Y=1]+P[X=1,Y=2]+P[X=1,Y=3] \\ &= 1/12+1/18=5/36 \end{aligned}$$

Similarly,

$$\begin{aligned} P[X=2] &= 19/36 \\ P[X=3] &= 1/3 \\ P[Y=1] &= 1/4 \\ P[Y=2] &= 14/45 \\ P[Y=3] &= 79/180 \end{aligned}$$

Problem 6 Solution

a)

$$\begin{aligned} F_W(W \le w) &= \int_0^\infty \int_0^{w/y} e^{-x} 3e^{-3y} dx dy \\ &= \int_0^\infty 3e^{-3y}(1-e^{-w/y}) dy \\ &= \int_0^\infty 3e^{-3y} - 3e^{-((3y^2+w)/y)}) dy \end{aligned}$$

Using Leibinitz theorem,

$$f_W(w) = \int_0^\infty (3/y) e^{-(3y^2-w)/y} dy$$

b) $W = X + Y$ Using the rule for addition of rvs,

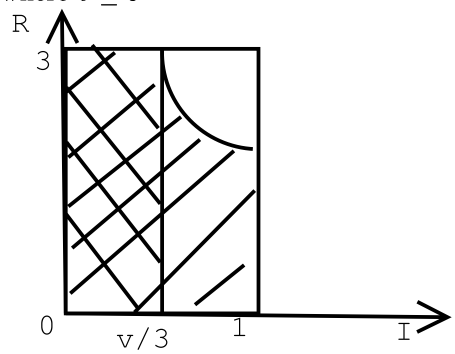
$$
\begin{aligned}
f_Z(z) &= f_X(x) * f_Y(y) \\
&= \int_{-\infty}^{\infty} f_X(v) f_Y(z-v) dv \\
&= \int_0^z e^{-v} 3e^{-3z+3v} dv \\
&= 3e^{-3z} \int_0^z e^{2v} dv \\
&= 3e^{-3z} \frac{1}{2}(e^{2z}-1) \\
&= \frac{3}{2}(e^{-z} - e^{-3z}) \;\; z \geq 0
\end{aligned}
$$

The limit of integration is 0 to z because it is the range over which both $f_X(v)$ and $f_Y(z-v)$ is nonzero.

Problem 7 Solution

$$
\begin{aligned}
F_V(v) &= P[IR \leq v] = P[R \leq v/I] \\
&= \int_0^3 \int_0^{v/3} 2i(r^2/9) di dr + \int_{v/3}^1 \int_0^{v/i} 2i(r^2/9) dr di
\end{aligned}
$$

where $v \leq 3$

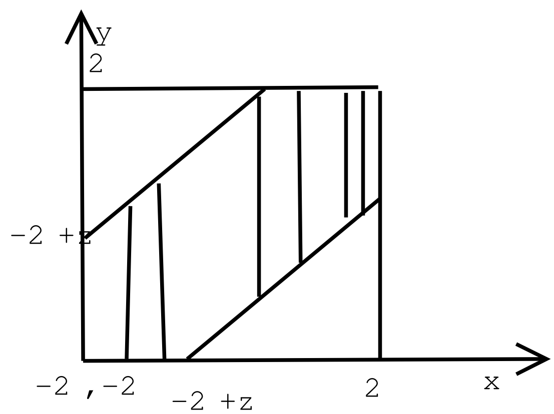


Differentiate both sides using Leibinitz rule,

$$
\begin{aligned}
f_V(v) &= \frac{1}{3}\int_0^3 2(\frac{v}{3})(\frac{r^2}{9}) dr + \int_{\frac{v}{3}}^1 (2i)(\frac{v}{i})^2 \frac{1}{9i} dr - \frac{1}{3}\int_0^3 2(\frac{v}{3})(\frac{v^2}{9}) dv \\
&= \int_{v/3}^1 (\frac{2}{9})(\frac{v}{i})^2 di \\
&= \begin{cases} \frac{2}{9}v^2(\frac{3}{v}-1) & 0 \leq v \leq 3 \\ 0 & elsewhere \end{cases}
\end{aligned}
$$

Problem 8 Solution

$$\begin{aligned} P[Z \le z] &= P[|X-Y| \le z] \\ &= P[-z \le X - Y \le z] \end{aligned}$$



The region $\{-z \le X - Y \le z\} = \{-z \le X - Y\} \cap \{X - Y \le z\} = \{Y \le X + z\} \cap \{Y \ge X - z\}$ which is the shaded region in the figure.

This can be calculated by using the diagram shown above.

$$F(z) = P[Z \le z] = \frac{1}{16} Area\, of\, shaded\, region = \frac{1}{16}(16 - (4-z)^2))$$

Therefore,

$$\begin{aligned} f_Z(z) &= \frac{1}{16}(2(4-z)) = \frac{1}{8}(4-z) \;\; for\, 0 \le z < 4 \\ & \quad 0 \qquad otherwise \end{aligned}$$

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