EE464: HOMEWORK 5 SOLUTIONS

Problem 1 Solution

X : Gaussian, $Y = aX + b$ is a linear combination of gaussian. Therefore Y is also gaussian.

$$\begin{aligned} E[Y] &= aE[X] + b = am + b = m' \\ Var[Y] &= a^2 Var[X] = a^2\sigma^2 = \sigma'^2 \end{aligned}$$

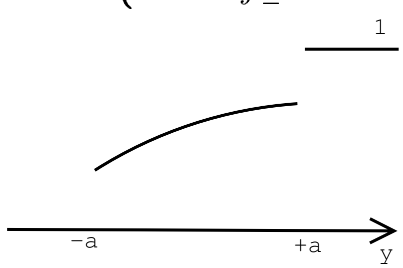
Therefore,

$$a = \sigma'/\sigma, \quad b = m' - am = m' - m\sigma'/\sigma$$

Problem 2 Solution

a)

$$F_Y(x) = \begin{cases} 0 & y < -a \\ F_X(x) & -a \le y \le a \\ 1 & y \ge a \end{cases}$$



From sketch of $F_Y(y)$ we see that

$$\begin{aligned} f_Y(y) &= F_Y'(y) = F_X(-a)\delta(y+a) + f_X(y) + (1 - F_Y(a))\delta(y-a) \; for \; |y| \le a \\ &= 0 \; otherwise \end{aligned}$$

b)

$f_X(x) = \frac{\beta}{2}e^{-\beta|x|}$ then, taking integration, we obtain

$$F_X(x) = \begin{cases} \frac{1}{2}e^{-\beta x} & x < 0 \\ 1 - \frac{1}{2}e^{-\beta x} & x \ge 0 \end{cases}$$

so,

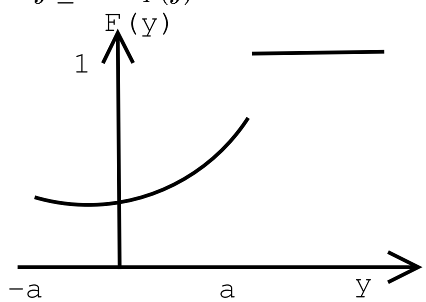
$$F_Y(y) = \begin{cases} 0 & y < -a \\ \frac{1}{2}e^{-\beta y} & -a \le y < 0 \\ 1 - \frac{1}{2}e^{-\beta y} & 0 \le y < a \\ 1 & y \ge a \end{cases}$$

and $f_Y(y) = \frac{1}{2}e^{-\beta a}\delta(y+a) + \frac{\beta}{2}e^{-\beta|y|} + \frac{1}{2}e^{-\beta a}\delta(y-a)$ for $|y| \le a$ and $f_Y(y) = 0$ elsewhere.

c) For $y < -a$, $F_Y(y) = 0$
For $-a \le y < a$:

$$F_Y(y) = \int_{-\infty}^{a} f_X(x)dx + \int_{a}^{y} f_X(x)dx$$

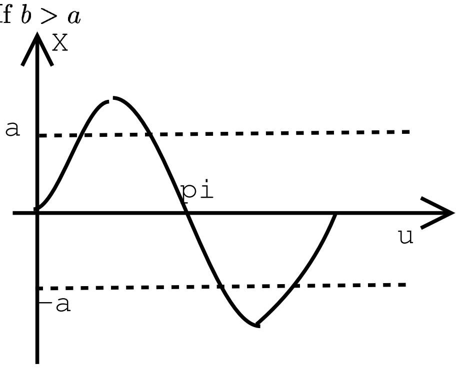
For $y \ge a$: $F_Y(y) = 1$



For $-a \le y \le a$,

$$f_Y(y) = \int_{-\infty}^{a} f(x)dx.\delta(y+a) + f_X(y) + \int_{a}^{\infty} f_X(x)dx.\delta(y-a)$$

d) If $b \le a$,see Example 3.28 on page 125.
If $b > a$



For $y < -a$, $f_Y(y) = 0$
For $y > a$, $f_Y(y) = 0$
For $-a < y < a$, $f_Y(y) = \frac{1}{\pi\sqrt{1-y^2}}$

$$P[Y = a] = P[Y = -a] = 1 - \frac{2}{\pi}\sin^{-1}\frac{a}{b}$$

Therefore,

$$\begin{aligned} f_Y(y) &= (1 - \frac{2}{\pi}\sin^{-1}\frac{a}{b})\delta(y+a) + \frac{1}{\pi\sqrt{1-y^2}} \\ &\quad +(1 - \frac{2}{\pi}\sin^{-1}\frac{a}{b})\delta(y-a)\ for\ |y| \le a \\ F_Y(y) &= \int_{-\infty}^{y} f_Y(t)dt \end{aligned}$$

Problem 3 Solution
a) For $y \leq 0$ $P[Y \leq y] = 0$
For $y > 0$ $P[Y \leq y] = P[e^X \leq y] = P[X \leq \ln y] = F_X(\ln y)$

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ F_X(\ln y) & y > 0 \end{cases}$$

For $y > 0$

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}F_Y(y) = F'_X(\ln y)\frac{1}{y} \\ &= \frac{1}{y}f_X(\ln y) \end{aligned}$$

b) If X is a Gaussian random variable, then,

$$f_Y(y) = \begin{cases} 0 & y \leq 0 \\ \frac{e^{-(\ln y - m)^2/2\sigma^2}}{y\sqrt{2\pi}\sigma} & y > 0 \end{cases}$$

Problem 4 Solution
a)

$$E[X] = \sum_{k=0}^{\infty} k\frac{\alpha^k}{k!}e^{-\alpha} = \alpha\sum_{k=1}^{\infty}\frac{\alpha^{k-1}}{(k-1)!}e^{-\alpha} = \alpha\sum_{k'=0}^{\infty}\frac{\alpha^{k'}}{k'!}e^{-\alpha} = \alpha$$

$$\begin{aligned} E[X^2] &= \sum_{k=0}^{\infty} k^2\frac{\alpha^k}{k!}e^{-\alpha} = \alpha\sum_{k=1}^{\infty} k\frac{\alpha^{k-1}}{(k-1)!}e^{-\alpha} \\ &= \alpha\sum_{k'=0}^{\infty}(k'+1)\frac{\alpha^{k'}}{k'!}e^{-\alpha} = \alpha(\alpha+1) \\ \sigma^2 &= E[X^2] - E^2[X] \\ &= \alpha \end{aligned}$$

b) $E[X] = \alpha = \lambda t = arrival\, rate * time$

Problem 5 Solution
$E[X] = \int_{-\infty}^{0} x\frac{1}{\pi(1+x^2)}dx + \int_0^{\infty}\frac{x}{\pi(1+x^2)}dx$
Consider the second term,

$$\int_0^y \frac{x}{\pi(1+x^2)}dx = \frac{1}{2\pi}\ln(1+x^2)|_0^y = \frac{\ln(1+y)}{2\pi} \to \infty$$

Therefore the integral does not exist, and so $E[X]$ does not exist.

Problem 6 Solution
X $U(0,1)$

$$\begin{aligned} F_Y(y) &= P[Y \le y] = P[-\ln X \le y] = P[\ln X \ge -y] \\ &= P[X \ge e^{-y}] = 1 - P[X < e^{-y}] = 1 - P[X \le e^{-y}] \\ &= \begin{cases} 1 - e^{-y} & y > 0 \\ 0 & y \le 0 \end{cases} \end{aligned}$$

This is because X is uniformly distributed and $P[X \le k] = k$ for $0 < k < 1$
To find density,

$$f_Y(y) = F'(y) = \begin{cases} e^{-y} & y > 0 \\ 0 & y \le 0 \end{cases}$$

Therefore Y is exponentially distributed.

Problem 7 Solution

$$F_X(x) = \int_0^x e^{-t} dt = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \ge 0 \end{cases}$$

$$\begin{aligned} F_Y(y) &= P[Y \le y] = P[e^{-X} \le y] \\ &= P[-X \le \ln y] = P[X \ge -\ln y] = 1 - F_X(-\ln y) \\ &= \begin{cases} e^{\ln y} = y & 0 < y < 1 \\ 0 & y \le 0 \\ 1 & y \ge 1 \end{cases} \end{aligned}$$

Therefore Y is a uniform r.v.

Problem 8 Solution

$$Y = X^2$$

$$\begin{aligned} f_Y(y) &= [f_X(\sqrt{y}) + f_X(-\sqrt{y})] \left| \frac{-1}{2\sqrt{y}} \right| \\ &= \frac{1}{\sqrt{2\pi y}} e^{-y/2} \, for \, y > 0 \end{aligned}$$

This is a chi-square random variable with 1 degree of freedom. See Example 3.25 and 3.26 from the textbook for another way of getting the solution.

Problem 9 Solution

$$\begin{aligned} F_Y(y) &= P[Y \le y] = P[|X| \le y] \\ &= P[-y \le X \le y] \\ &= \begin{cases} y & 0 < y < 1 \\ 0 & y \le 0 \\ 1 & y > 0 \end{cases} \end{aligned}$$

Therefore Y is a uniform random variable $Y\ U[0,1)$.

Problem 10 Solution
a)

$$\begin{aligned} E[X] &= \sum_{k=1}^{\infty} k\beta(1-\beta)^{k-1} = \beta \sum_{k=1}^{\infty} k(1-\beta)^{k-1} \\ &= \beta(-\frac{d}{d\beta}\sum_{k=0}^{\infty}(1-\beta)^k) \; ...Verify! \\ &= \beta(-\frac{d}{d\beta}(\frac{1}{\beta})) = \frac{1}{\beta} \end{aligned}$$

b)

$$\begin{aligned} E[X^2] &= \sum_{k=1}^{\infty} k^2\beta(1-\beta)^{k-1} = \beta \sum_{k=1}^{\infty} k^2(1-\beta)^{k-1} \\ &= \beta(1-\beta)\sum_{k=1}^{\infty} k^2(1-\beta)^{k-2} \\ &= \beta(1-\beta)\sum_{k=1}^{\infty} k(k-1)(1-\beta)^{k-2} + \sum_{k=1}^{\infty} k\beta(1-\beta)^{k-1} \\ &= \beta(1-\beta)(\frac{2}{\beta^3}) + \frac{1}{\beta} = \frac{2-\beta}{\beta^2} \end{aligned}$$

Here we used the fact that, $\sum_{k=1}^{\infty} k(k-1)(1-\beta)^{k-2} = \frac{d^2}{d\beta^2}\sum_{k=0}^{\infty}(1-\beta) = \frac{d^2}{d\beta^2}(\frac{1}{\beta}) = 2/\beta^3$

Note that for infinite series, $\frac{d}{dx}\sum f(x) = \sum f'(x)$ is **not** always true. But in this case it has been proved to be true.

$$\begin{aligned} Var(X) &= E[X^2] - E[X]^2 \\ &= \frac{2-\beta}{\beta^2} - \frac{1}{\beta^2} \\ &= \frac{1-\beta}{\beta^2} \end{aligned}$$

Problem 11 Solution

$$\begin{aligned} E[X] &= \int_0^{\infty} x(2e^{-2x})dx \\ &= \int_0^{\infty} (y/2)(e^{-y})dy \\ &= \frac{1}{2}\Gamma(2) = \frac{1}{2}1! \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} E[X^2] &= \int_0^\infty x^2(2e^{-2x})dx \\ &= \frac{1}{4}\int_0^\infty y^2(e^{-y})dy \\ &= \frac{1}{4}\Gamma(3) = \frac{1}{4}2! \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} Var(X) &= E[X^2] - E[X]^2 \\ &= \frac{1}{2} - \frac{1}{4} \\ &= \frac{1}{4} \end{aligned}$$

Problem 12 Solution
Using $E[g(X)] = \int g(x)f_X(x)dx$

$$\begin{aligned} E[Y] &= E[X^4] = \int_0^1 x^4 3x^2 dx \\ &= 3[x^7/7]_0^1 \\ &= \frac{3}{7} \end{aligned}$$

Similarly,

$$\begin{aligned} E[Y^2] &= E[X^8] = \int_0^1 x^8 3x^2 dx \\ &= 3[x^{11}/11]_0^1 \\ &= \frac{3}{11} \end{aligned}$$

$$\begin{aligned} Var(Y) &= E[Y^2] - E[Y]^2 \\ &= \frac{3}{11} - \frac{9}{49} \\ &= 0.089 \end{aligned}$$

Problem 13 Solution
a) Using Tchebycheff's inequality,

$$P[|X-\mu| \geq \frac{3}{2}\sigma] \leq (\frac{2}{3})^2 = 4/9 = 0.4444$$

b)

$$\begin{aligned} P[|X-1| &\geq \frac{3}{2}(\frac{1}{3})] = P[X-1\geq 0.5]+P[1-X\geq 0.5] = P[X\geq 1.5]+P[X\leq 0.5] \\ &= \frac{\sqrt{3}}{2}((1+1/\sqrt{3})-1.5)+\frac{\sqrt{3}}{2}(0.5-(1-1/\sqrt{3})) = 0.067+0.067 = 0.134 \end{aligned}$$

Thus we can see that the inequality is not really tight in this particular case.

Problem 14 Solution

$$\begin{aligned} M_X(s) &= E[e^{sX}] \\ &= \frac{1}{b-a}\int_a^b e^{sx}.1dx \\ &= \frac{1}{b-a}\left.\frac{e^{sx}}{s}\right|_a^b \\ &= \frac{e^{sb}-e^{sa}}{s(b-a)} \end{aligned}$$

This exists when $s \neq 0$.

Problem 15 Solution

$$\begin{aligned} M_X(s) &= \sum_{k=1}^{\infty} e^{sk}p(1-p)^{k-1} \\ &= e^s p \sum_{k=1}^{\infty} e^{s(k-1)}(1-p)^{k-1} \\ &= e^s p \sum_{k=1}^{\infty} e^{s(k-1)}(1-p)^{k-1} \\ &= \frac{e^s p}{1-e^s(1-p)} \end{aligned}$$

The condition for convergence of geometric series is $|e^s(1-p)| < 1$. If $s = \alpha + j\beta$, then $|e^s(1-p)| = e^{\alpha}(1-p) < 1 \Leftrightarrow Re\{s\} = \alpha < -\ln(1-p)$. Therefore condition for existence of $M_X(s)$ is $Re\{s\} < -\ln(1-p)$.

$$\begin{aligned} M'(s) &= \frac{(1-e^s(1-p))pe^s + pe^s e^s(1-p)}{(1-e^s(1-p))^2} \\ E[X] &= M'(0) = \frac{p^2+p-p^2}{p^2} \\ &= \frac{1}{p} \end{aligned}$$

Similarly,

$$
\begin{aligned}
M'(s) &= \frac{(1-e^s(1-p))pe^s + pe^s e^s(1-p)}{(1-e^s(1-p))^2} \\
&= \frac{pe^s}{(1-e^s(1-p))^2} \\
M''(s) &= \frac{(1-e^s(1-p))^2 pe^s + pe^s 2(1-p)e^s(1-e^s(1-p))}{(1-e^s(1-p))^4} \\
E[X^2] &= M''(0) = \frac{p^2 p + 2p^2(1-p)}{p^4} \\
&= \frac{2-p}{p^2}
\end{aligned}
$$

$$
\begin{aligned}
Var(X) &= E[X^2] - E[X]^2 \\
&= \frac{2-p}{p^2} - \frac{1}{p^2} \\
&= \frac{1-p}{p^2}
\end{aligned}
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