EE464: HOMEWORK 4 SOLUTIONS

Problem 1. Solution

a) $1 = \int_{-1}^{1} f(x)dx = c\int_{-1}^{1}(1-x^4)dx = c[x - x^5/5]_{-1}^{1} = 8c/5$
Therefore $c = 5/8$
b) $F(x) = 0$ for $x < -1$; $F(x) = 1$ for $x > 1$
For $-1 \le x \le 1$

$$\begin{aligned} F_X(x) &= \int_{-1}^{x}(5/8)(1-y^4)dy \\ &= (1/2) + (1/8)(5x - x^5) \end{aligned}$$

c) $P[|X| < 1/2] = P[-1/2 < X < 1/2] = F_X(1/2) - F_X(-1/2) = \frac{79}{8(16)} = 0.6172$

Problem 2. Solution

a) $f_X = \begin{cases} 0 & |x| > a \\ c(1 - \frac{|x|}{a}) & |x| \le a \end{cases}$

$$1 = \int_{-a}^{a} f_X(x)dx = Area\, of\, triangle = c(2a)/2 = ac$$

Therefore $c = 1/a$
b) $F_X(x) = 0$ for $x < -a$; $F_X(x) = 1$ for $x > a$
For $-a \le x \le 0$

$$\begin{aligned} F_X(x) &= \frac{1}{a}\int_{-a}^{x}(1 + (y/a))dy \\ &= \frac{1}{2} + \frac{1}{a}(x + x^2/2a) \end{aligned}$$

For $0 \le x \le a$

$$\begin{aligned} F_X(x) &= \int_{-a}^{0} f(y)dy + \int_{0}^{a} f(y)dy \\ &= \frac{1}{2} + \frac{1}{a}\int_{0}^{x}(1 + y/a)dy \\ &= \frac{1}{2} + \frac{1}{a}(x - x^2/2a) \end{aligned}$$

c) $P[|X| < b] = \frac{1}{2} = F_X(b) - F_X(-b) = \frac{2}{a}(b - \frac{b^2}{2a})$
Therefore $b = a(1 - \frac{1}{\sqrt{2}})$

Problem 3. Solution

a) Let I_k denote the outcome of the kth Bernoulli trials. The probability that the single event occured the the kth trial is:

$$\begin{aligned} P[I_k = 1|X = 1] &= \frac{P[I_k = 1\, and\, I_j = 0\, for\, all\, j \neq k]}{P[X = 1]} \\ &= \frac{P[00...10..0]}{P[X = 1]} \\ &= \frac{p(1-p)^{n-1}}{\binom{n}{1} p(1-p)^{n-1}} \\ &= 1/n \end{aligned}$$

Thus the single event is equally likely to have occured in any of the n^{th} trials.
b) The probability that the two successes occurred in trials j and k is:

$$\begin{aligned} P[I_j = 1, I_k = 1|X = 2] &= \frac{P[I_j = 1, I_k = 1, I_m = 0\, for\, all\, m \neq j, k]}{P[X = 2]} \\ &= \frac{p^2(1-p)^{n-2}}{\binom{n}{2} p^2(1-p)^{n-2}} \\ &= \frac{1}{\binom{n}{2}} \end{aligned}$$

Thus all $\binom{n}{2}$ possibilities for j and k are equally likely.

c) If $X = k$ then location of successes selected at random from among the $\binom{n}{k}$ possible permutations.

Problem 4. Solution

a)

$$\begin{aligned} P[N > k] &= \sum_{j=k+1}^{\infty} p(1-p)^{j-1} = p(1-p)^k \sum_{i=0}^{\infty}(1-p)^i \\ &= p(1-p)^k \frac{1}{1-(1-p)} = (1-p)^k \end{aligned}$$

b)

$$\begin{aligned} F_N(x) &= P(N \leq x) = \sum_{j=1}^{[x]} p(1-p)^{j-1} = p\frac{1-(1-p)^{[x]}}{1-(1-p)} \\ &= 1-(1-p)^{[x]} \; for\, x \geq 0 \end{aligned}$$

c)

$$\begin{aligned} P[N\, is\, even] &= \sum_{j=1}^{\infty} p(1-p)^{2j} \\ &= p(1-p)\frac{1}{1-(1-p)^2} \\ &= \frac{1-p}{2-p} \end{aligned}$$

d)

$$\begin{aligned} P[M = k|M \leq m] &= \frac{P[M = k, M \leq m]}{P[M \leq m]} = \frac{P[M = k]}{P[M \leq m]} \\ &= \frac{(1-p)^{k-1}p}{1-(1-p)^m} \; for\, 1 \leq k \leq m \end{aligned}$$

Problem 5. Solution

$$\begin{aligned} P[error|v = -1] &= P[Y \geq 0|v = -1] \\ &= P[-1 + N \geq 0] \\ &= P[N \geq 1] = Q(1) \\ &= 0.159 \end{aligned}$$

$$\begin{aligned} P[error|v=+1] &= P[Y<0|v=1] \\ &= P[1+N<0] \\ &= P[N<-1] = 1-Q(-1) \\ &= 0.159 \end{aligned}$$

Problem 6. Solution

a)

$$f(x) = \begin{cases} 20x^3(1-x) & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

i) $f(x) \ge 0$ is true for all x.

ii)

$$\begin{aligned} \int_0^1 20x^3(1-x)dx &= 20[(x^4/4)-(x^5/5)]_0^1 \\ &= 1 \end{aligned}$$

Therefore $f(x)$ is a valid pdf.

b) When $0 \le x \le 1$,

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(y)dy \\ &= \int_0^x 20x^3(1-x)dx \\ &= 5x^4 - 4x^5 \end{aligned}$$

Therefore,

$$F(x) = \begin{cases} 0 & x < 0 \\ 5x^4 - 4x^5 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

c)

$$\begin{aligned} P(X \le 2/3) &= F(2/3) \\ &= 5(2/3)^4 - 4(2/3)^5 \\ &= 0.461 \end{aligned}$$

d)

$$\begin{aligned} P(X < 2/3|\frac{1}{4} < X < \frac{3}{4}) &= \frac{P(\{X < 2/3\} \cap \{\frac{1}{4} < X < \frac{3}{4}\})}{P(\frac{1}{4} < X < \frac{3}{4})} \\ &= \frac{P(\frac{1}{4} < X < \frac{2}{3})}{P(\frac{1}{4} < X < \frac{3}{4})} \\ &= \frac{F(\frac{2}{3}) - F(\frac{1}{4})}{F(\frac{3}{4}) - F(\frac{1}{4})} \\ &= \frac{0.461 - 0.0156}{0.6328 - 0.0156} \\ &= 0.7215 \end{aligned}$$

Problem 7. Solution

a)

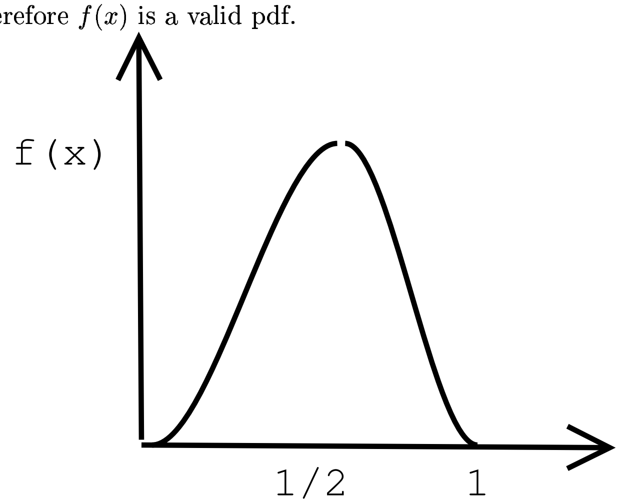
$$f(x) = \begin{cases} 6x(1-x) & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

i) $f(x) \ge 0$ is true for all x.

ii)

$$\begin{aligned} \int_0^1 6x(1-x)dx &= 6[(x^2/2) - (x^3/3)]_0^1 \\ &= 1 \end{aligned}$$

Therefore $f(x)$ is a valid pdf.



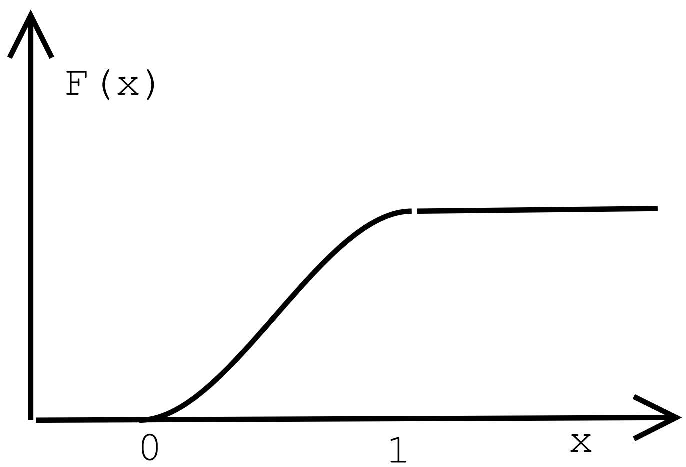
b) To find distribution function,

When $0 \le x \le 1$,

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(y)dy \\ &= \int_{0}^{x} 6x(1-x)dx \\ &= 3x^2 - 2x^3 \end{aligned}$$

Therefore,

$$F(x) = \begin{cases} 0 & x < 0 \\ 3x^2 - 2x^3 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$



c)

$$\begin{aligned} P(X < b) &= 2P(X > b) \\ F(b) &= 2(1 - F(b)) \\ F(b) &= 2/3 \end{aligned}$$

Therefore

$$\begin{aligned} 3b^2 - 2b^3 &= 2/3 \\ b &= 0.6130 \end{aligned}$$

Problem 8. Solution

a)

$$
\begin{aligned}
P(X \leq 2) &= \int_0^2 3e^{-3x} dx \\
&= 1 - e^{-6} \\
&= 0.9975
\end{aligned}
$$

b)

$$
\begin{aligned}
P(X \leq 2 | X > 1) &= \frac{P(1 < X \leq 2)}{P(X > 1)} \\
&= \frac{\int_1^2 3e^{-3x} dx}{\int_1^\infty 3e^{-3x} dx} \\
&= \frac{e^{-3} - e^{-6}}{e^{-3}} \\
&= 0.9502
\end{aligned}
$$

Problem 9. Solution

$$
\begin{aligned}
P(Y \leq 2/3) &= P(\{Y \leq 2/3\}|H)P(H) + P(\{Y \leq 2/3\}|T)P(T) \\
&= F_{U[0,1]}(2/3)(1/2) + F_{U[0,2]}(2/3)(1/2) \\
&= (2/3)(1/2) + (1/3)(1/2) \\
&= 1/2
\end{aligned}
$$