EE464: HOMEWORK 3 SOLUTIONS

Problem 1. Solution

Let k be the number of defective items in a batch of n tested items, then k has binomial probabilities with parameters n and $p = \frac{1}{10}$. Therefore,

$$\begin{aligned} P[k > 1] &= 1 - P[k \le 1] = 1 - P[k = 0] - P[k = 1] \\ &= 1 - (1-p)^n - n(1-p)^{n-1}p \\ &= 1 - \left(\frac{9}{10}\right)^n - n\left(\frac{9}{10}\right)^{n-1}\left(\frac{1}{10}\right) \end{aligned}$$

Problem 2. Solution

Let $p =$ probability of no defect $= \frac{95}{100} = \frac{19}{20}$. And let n be the total number of chips purchased and k be number of chips with no defect.

Pick n such that $P[k \ge 10] \ge 0.9$

$$P[k \ge 10] = \sum_{k=10}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

for $n = 11$ $P[k \ge 10] = 0.89811$

for $n = 12$ $P[k \ge 10] = 0.98093$

Therefore pick $n = 12$.

Problem 3. Solution

a) Let m be the number of tails $0 \le m \le n$. Then $n - m$ is the number of heads and the difference is $Y = n - m - m = n - 2m$, $0 \le m \le n$.

$$S_Y = \{-n, -n+2, ..., n-2, n\}$$

b) $\{Y = 0\}$ iff $\{m = n - m\}$ iff $\{m = \frac{n}{2}\}$

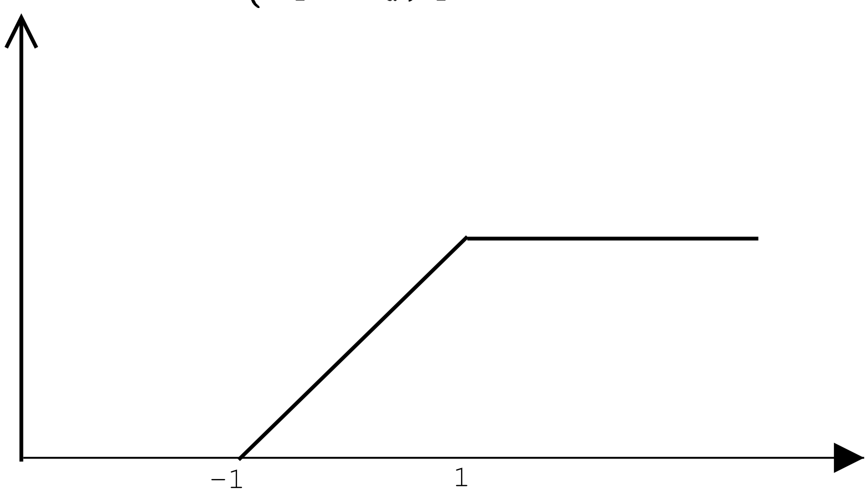
c) $\{Y \le k\}$ iff $\{n - 2m \le k\}$ iff $\{\frac{n-k}{2} \le m\}$

Problem 4. Solution

a) $S_Y = \{Y : 0 \leq Y < 1\}$
b) $\{Y \leq y\} = \{\text{point is inside circle of radius } y\}$
c) $P[Y \leq y] = \frac{\pi y^2}{\pi 1^2} = y^2$

Problem 5. Solution

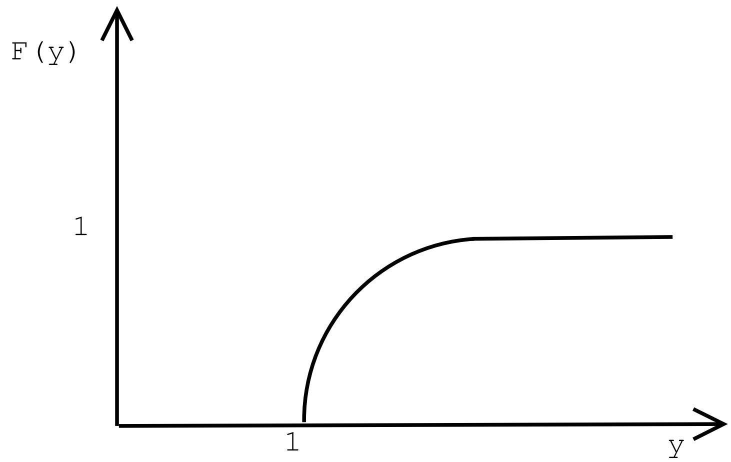
$$F_U(u) = P[U \leq u] = \begin{cases} 0 & u < -1 \\ \frac{u+1}{2} & -1 \leq u \leq 1 \\ 1 & u > 1 \end{cases}$$



$$\begin{aligned}
P[U > 0] &= 1 - P[U \leq 0] = 1 - F_U(0) = \frac{1}{2} \\
P[U < 5] &= 1 \\
P[|U| < \frac{1}{3}] &= P[-\frac{1}{3} < U < \frac{1}{3}] = F_U(\frac{1}{3}) - F_U(-\frac{1}{3}) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \\
P[\frac{1}{3} < U < \frac{1}{2}] &= F_U(1/2) - F_U(1/3) = (3/4) - (2/3) = 1/12 \\
P[|U| \geq \frac{3}{4}] &= 1 - P[|U| < \frac{3}{4}] = 1 - (F_U(3/4) - F_U(-3/4)) = 1 - 6/8 = 1/4
\end{aligned}$$

Problem 6. Solution

a) Plot of cdf is shown in the figure



b) $P[k < Y \leq k+1] = F_Y(k+1) - F_Y(k) = (1/k)^{-n} - (1/(k+1))^{-n}$

Problem 7. Solution

Since $\sigma \geq 0$,

$P[\sigma \leq R \leq 2\sigma] = P[\sigma < R \leq 2\sigma] = F_R(2\sigma) - F_R(\sigma) = e^{-1/2} - e^{-2}$
$P[R > 3\sigma] = 1 - P[R \leq 3\sigma] = 1 - F_R(3\sigma) = e^{-9/2}$

Problem 8. Solution

a) If f and g are pdfs defined on the same interval $[a, b]$, then,

$$\begin{aligned} \int_a^b (f(x) + g(x))dx &= \int_a^b f(x)dx + \int_a^b g(x)dx \\ &= 1 + 1 \\ &= 2 \neq 1 \end{aligned}$$

Therefore $f(x) + g(x)$ is not a valid pdf.

b) For any number β, $s.t.\, 0 < \beta < 1$

(i) $\beta f(x) + (1-\beta)g(x) \geq 0$ for all values $a \leq x \leq b$.

(ii)

$$\begin{aligned} \int_a^b (\beta f(x) + (1-\beta)g(x))dx &= \beta \int_a^b f(x)dx + (1-\beta)\int_a^b g(x)dx \\ &= \beta + (1-\beta) \\ &= 1 \end{aligned}$$

Therefore $\beta f(x) + (1-\beta)g(x)$ is a valid pdf on the interval $[a, b]$.

Problem 9. Solution

The roots of equation $4x^2 + 4Kx + K + 2 = 0$ are real if the discriminant $b^2 - 4ac \geq 0$. The discriminant for this quadratic equation is given by,

$$\begin{aligned} b^2 - 4ac &= 16K^2 - 16(K+2) \\ &= 16K^2 - 16K - 32 \end{aligned}$$

Therefore, the condition is $16K^2 - 16K - 32 \geq 0 \Leftrightarrow K^2 - K - 2 \geq 0$.

The roots of equation $K^2 - K - 2 = 0$ are $\frac{1 \pm \sqrt{1+8}}{2}$,which are $+2$ and -1.

Therefore, $K^2 - K - 2$ is ≥ 0 over $[2, \infty)$ and $(-\infty, -1]$. However K is uniformly distributed over $[0, 5]$. Therefore the condition for real roots is $2 \leq K \leq 5$.

And

$$\begin{aligned} P[real\, roots] &= P[2 \leq K \leq 5] \\ &= \frac{5-2}{5} \\ &= 0.6 \end{aligned}$$

Problem 10. Solution

a) Since summation of mass function should be 1,

$$\begin{aligned} 1 &= \sum_{n=1}^{\infty} P(X = n) \\ &= k \sum_{n=1}^{\infty} (1-\beta)^{n-1} \\ &= k/\beta \end{aligned}$$

Therefore $k = \beta$.

b) Since $0 < \beta < 1$, $0 < 1 - \beta < 1$. Therefore $P[X = r] = \beta(1-\beta)^{r-1}$ is monotonically decreasing function of r. Therefore by choosing the smallest possible value of r, we have $r = 1$ is the mode.

Problem 11. Solution

a) Lets say F_1= 1st coin is fair, and F_2= second coin is fair.

$$P[X = 0, Y = 0] = 0$$

This is because getting both tails is impossible.

On the other hand,

$$\begin{aligned} P[X=0] &= P[X=0|F_1]P[F_1]+P[X=0|F_2]P[F_2] \\ &= (1/2)(1/2)+0 \\ &= 1/4 \end{aligned}$$

Similarly,

$$\begin{aligned} P[Y=0] &= P[Y=0|F_1]P[F_1]+P[Y=0|F_2]P[F_2] \\ &= 0+(1/2)(1/2) \\ &= 1/4 \end{aligned}$$

Now,

$$P[X=0,Y=0]=0 \quad \neq \quad P[X=0]P[Y=0]=1/16$$

Therefore X and Y are not independent.
b) Let $X=0$ if the outcome is tail and $X=1$ if the outcome is head.
Let F denote fair coin is tossed.

$$\begin{aligned} P[X=1] &= P[X=1|F]P[F]+P[X=1|F^c]P[F^c] \\ &= (1/2)(1/2)+(1)(1/2) \\ &= 3/4 \end{aligned}$$

Also $P[X=1|F]=1/2$ and $P[F]=1/2$
Now by using Bayes rule,

$$\begin{aligned} P[F|(X=1)] &= \frac{P[X=1|F]P[F]}{P[X=1]} \\ &= \frac{(1/2)(1/2)}{3/4} \\ &= 1/3 \end{aligned}$$

Therefore $P[F]=1/3$ and $P[F^c]=2/3$. Therefore the coin which is not tossed is more likely to be the fair coin.