EE464: HOMEWORK NO 2 SOLUTIONS

Problem 1. Solution

$P[\{a,c\}] = P[\{a\}] + P[\{c\}] = 5/8$

$P[\{b,c\}] = P[\{b\}] + P[\{c\}] = 7/8$

$P[\{a,b,c\}] = P[S] = 1 = P[\{a\}] + P[\{b\}] + P[\{c\}]$

$\Longrightarrow\ P[\{a\}] = 1/8,\ P[\{b\}] = 3/8,\ P[\{c\}] = 4/8$

Problem 2. Solution

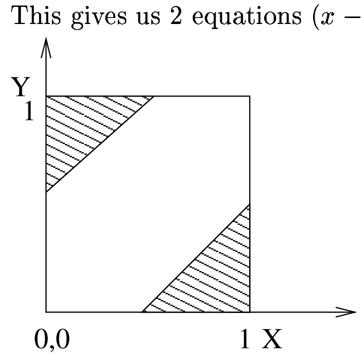
Let 1st pt is x and second pt is y.

The sample space is shown in the figure below.

The condition we are interested in is $|x - y| > 1/2$

$\Leftrightarrow -1/2 > (x - y) > 1/2$

This gives us 2 equations $(x - y) > 1/2$ *and* $(x - y) < -1/2$



This is shaded area in the figure.

$P\{|x - y| > 1/2\} = \frac{shaded\,area}{total\,area} = 1/4.$

Problem 3. Solution

The order in which 4 toppings are selected does not matter. So we have sampling *without ordering.*

If the toppings *may not be repeated*, thenwe can have
$\begin{pmatrix} 15 \\ 4 \end{pmatrix} = 1365$ possible deluxe pizzas.

If the toppings may be repeated, then we have sampling *with replacement and without ordering.* The number of such arrangements is
$\begin{pmatrix} 14+4 \\ 4 \end{pmatrix} = 3060$ possible deluxe pizzas.

Problem 4. Solution

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!} = \begin{pmatrix} n \\ n-k \end{pmatrix}$$

Problem 5. Solution

a) By law of total probability,

$P[Y = 0] = P[Y = 0 \,|\, X = 0]P[X = 0] + P[Y = 0 \,|\, X = 1]P[X = 1]$
$+ P[Y = 0 \,|\, X = 2]P[X = 2]$

$= (1-\epsilon)(1/2) + 0(1/4) + \epsilon(1/4)$

$= \frac{1}{2} - \frac{\epsilon}{4}$

Similarly,

$P[Y = 1] = \epsilon(\frac{1}{2}) + (1-\epsilon)\frac{1}{4} + 0(\frac{1}{4}) = \frac{1}{4} + \frac{\epsilon}{4}$

$P[Y = 2] = 0(\frac{1}{2}) + (\epsilon)\frac{1}{4} + (1-\epsilon)(\frac{1}{4}) = \frac{1}{4}$

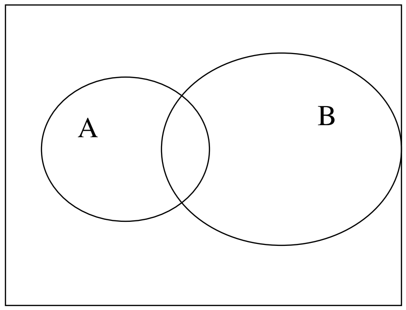
b)Using Bayes Rule,

$P[X = 0 \,|Y = 1] = \frac{P[Y=1 \,|X=0]P[X=0]}{P[Y=1]} = \frac{\frac{\epsilon}{2}}{\frac{1}{4}+\frac{\epsilon}{4}} = \frac{2\epsilon}{1+\epsilon}$

$P[X = 1 \,|Y = 1] = \frac{P[Y=1 \,|X=1]P[X=1]}{P[Y=1]} = \frac{(1-\epsilon)\frac{1}{4}}{\frac{1}{4}+\frac{\epsilon}{4}} = \frac{1-\epsilon}{1+\epsilon}$

$P[X = 2 \,|Y = 1] = 0$

Problem 6. Solution



Venn diagram

$P\{A \cup B\} = P\{A\} + P\{B\} - P\{A \cap B\} = x + y - z$

From the venn diagram we can see that

a)$P\{A^c \cup B^c\} = P\{(A \cap B)^c\} = 1 - P\{A \cap B\} = 1 - z$

b) $P\{A \cap B^c\} = P\{A\} - P\{A \cap B\} = x - z$

c) $P\{A^c \cup B\} = 1 - P\{A \cap B^c\} = 1 - (x - z)$

d) $P\{A^c \cap B^c\} = P\{(A \cup B)^c\} = 1 - P\{A \cup B) = 1 - x - y + z$

Problem 7. Solution

$P(2Redballs)$

a) Without replacement

$$P(2R) = \frac{\begin{pmatrix} 3 \\ 2 \end{pmatrix}\begin{pmatrix} 9 \\ 1 \end{pmatrix}}{\begin{pmatrix} 12 \\ 3 \end{pmatrix}} = \frac{27}{220} = 0.1227$$

b) With replacement

There are 3 possible sequences

1. $R R R^c$

2. $R R^c R$

3. $R^c R R$

In case 1 there are 3x3x9 ways. Similarly, for 2 and 3.

Therefore,

$P\{2R\} = \frac{(3\times3\times9)\times3}{12^3} = 0.1406$

a') Without replacement

$$P(2R,1W) = \frac{\binom{3}{2}\binom{5}{1}}{\binom{12}{3}} = \frac{15}{220} = .068$$

b') With replacement,

$P(2R,1W) = \frac{(3\times3\times5)\times3}{12^3} = 0.078$

Problem 8. Solution

a) The sample space $S = \{h, th, tth, ttth, tttth.....\}$

b) $P\{1^{st}\, head\, on\, k^{th}\, toss\} = P\{k-1\, successive\, tails \cap\; head\, on\, k^{th}\, toss\}$

$= (\frac{1}{2})^{k-1}\frac{1}{2} = \frac{1}{2^k}$

c) $P(\Omega) = P\{(1^{st}\, head\, on\, 1^{st}\, toss) \cup (1^{st}\, head\, on\, 2^{st}\, toss) \cup (..)\}$

$= P\{(1^{st}\, head\, on\, 1^{st}\, toss)\} + P\{(1^{st}\, head\, on\, 2^{st}\, toss)\} + ... + P\{(..)\}$

$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... = \sum_{i=1}^{\infty}\frac{1}{2^i} = 1$

d) $P(k > 4) = P(k = 5) + P(k = 6) + ... = \sum_{k=5}^{\infty}\frac{1}{2^k} = \frac{1}{2^5}(\frac{1}{(1-1/2)}) = \frac{1}{2^4} = 0.0625$

e)$P(k > 10) = P(k = 11) + P(k = 12) + ... = \sum_{k=11}^{\infty}\frac{1}{2^k} = \frac{1}{2^{11}}(\frac{1}{(1-1/2)}) = \frac{1}{2^{10}} = 0.0009766$

f) $P(B^c) = 1 - P(B) = 1 - 0.0009766 = 0.9990234$

g) $P(A \cap B) = P(B) = 0.0009766$

h) $P(A \cup B) = P(A) = 0.0625$

Problem 9. Solution

Total # of ways of choosing each objects $= N$

Total # of ways of choosing k objects $= N^k$

Now,

\# of ways choosing k objects from N objects such than no object is chosen more than once is given by

$N(N-1)(N-2)(N-3)..(N-k+1) = \frac{N!}{(N-k)!}$

Therefore,

$P(choosing\, k\, from\, N) = \frac{\frac{N!}{(N-k)!}}{N^k}$

Problem 10. Solution

a) First calculate

P{all persons have birthdates on different dates} $= \frac{365*364*...*(365-n+1)}{365^n} = \frac{365P_n}{365^n}$

P{atleast 2 persons have the same birthdate} = 1 - P{all persons have birthdates on different dates}

$=1 - \frac{365P_n}{365^n}$

b) for n=50, P= $1 - \frac{365P_{50}}{365^{50}} = 1 - 0.03 = 0.97$

c) If n=23, P{persons have different birthdates} = 0.493

therefore P{atleast 2 persons have the same birthdate} = 0.507

Problem 11. Solution

LHS = $P(AB|C) = \frac{P(ABC)}{P(C)}$ here by AB we mean $A \cap B$.

$= \frac{P(ABC)}{P(BC)} \frac{P(BC)}{P(C)} = P(A|BC)P(B|C)$ = RHS

Problem 12. Solution

$P(B|A) = \frac{P(A \cap B)}{P(A)} = 0.4$

therefore, $P(A \cap B) = P(B|A)P(A) = 0.4P(A)$ .....(1)

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.5 = P(A) + 0.2 - 0.4P(A)$

$\therefore 0.6P(A) = 0.3 \Rightarrow P(A) = 0.5$

Therefore from eq(1),

$P(A \cap B) = 0.4 * 0.5 = 0.2$

$P(A|B) = P(A \cap B)/P(B) = 0.2/0.2 = 1$

Problem 13. Solution

Suppose A and B are independent,

$P(A \cap B) = P(A)P(B) = P(\phi) = 0$

Therefore $A \cap B = \phi \implies either\, P(A) = 0\, or\, P(B) = 0$

Therefore the answer is Yes,

In case if $P(A), P(B) \neq 0$, then they cannot be independent.

Problem 14. Solution

Let R_2 be event that second ball picked is red.

Let R_1 be event that first ball picked is red.

Let B_1 be event that second ball picked is blue.

$P(R_2) = P(R_2|R_1)P(R_1) + P(R_2|B_1)P(B_1)$ by total probability,

$= (\frac{R+C}{R+B+C})(\frac{R}{R+B}) + (\frac{R}{R+B+C})(\frac{B}{R+B}) = (\frac{R(R+C+B)}{(R+B+C)(R+B)}) = \frac{R}{R+B}$

Problem 15. Solution

Let F be event that fair coin is given

Let U be event that unfair coin is given

$P(F) = P(U) = 1/2$ since the choice is made at random.

Let $H2$ be an event that 2 heads are obtained.

$P(H2|F) = (1/2)(1/2) = 1/4$ and

$P(H2|U) = (1/3)(1/3) = 1/9$

By using law of total probability, we get,

$P(H2) = P(H2|F)P(F) + P(H2|U)P(U) = \frac{1}{4}\frac{1}{2} + \frac{1}{9}\frac{1}{2} = 13/72$

Now using Bayes rule,

$P(F|H2) = \frac{P(H2|F)P(F)}{P(H2)} = \frac{(1/4)(1/2)}{(13/72)} = 0.6923$

Problem 16. Solution

Let A is the event that the subject has the disease.

B is the event that the test is positive.

a) $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = (0.98)(0.001) + (0.05)(0.999) = 0.05093$

b) $P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{(0.98)(0.001)}{0.05093} = 0.019$