EE 464 Final Exam Solutions

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Problem	Points	Score	Problem	Points	Score
1	4		11	4	
2	4		12	4	
3	4		13	4	
4	4		14	4	
5	4		15	4	
6	4		16	4	
7	4		17	4	
8	4		18	12	
9	4		19	9	
10	4		20	11	
Total	=>	=>	=>	100	

Instructions and Information:

1) Print your name and location at the top of the page.
2) Make sure your exam has 20 problems and 25 numbered pages. Pages 8-9 are for extra workspace.
3) This is a closed book exam. A formula sheet is provided at the end of the exam. You may use a calculator. **You have 2 hours to take this exam**.
4) Partial credit will be given but you must **show your work (when appropriate) to receive any credit**.
5) **Circle or box your final answers**.

Problem 1. Let $\Omega = \{u,\ v,\ w\}$. Find the subset of Ω that one would need to add (if any) to the following list of subsets in order that the list constitutes a sigma field.

$$\phi,\ \{u,\ v\},\ \{u,\ w\},\ \{v,\ w\},\ \{u\},\ \{v\},\ \{w\}.$$

ⓐ $\{u,\ v,\ w\}$ b. {uv, w} c. {u, vw} d. {uv, vw} e. None of these.

Solution. To make this a σ-field we need to add $\{u,\ v,\ w\}$.

Problem 2. An urn contains 3 red and 4 white balls. Three balls are chosen at random. Compute the probability of choosing 2 red balls if the sampling is done without replacement.

a. 0.6 ⓑ. 0.3 c. 0.5 d. None of these.

Solution.

$$P(2\ red) = \frac{\binom{3}{2}\binom{4}{1}}{\binom{7}{3}} = 0.34 \rightarrow 0.3.$$

Note: The answer to this problem is 0.3 to one decimal place but due to the fact that we usually write these numbers to two decimal places you got credit if your answer was d.

Problem 3. Compute the probability of a pair in poker (exactly 2 cards of equal face value and 3 cards of different face values).

a. $\dfrac{\binom{13}{1}\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}$ ⓑ $\dfrac{\binom{13}{1}\binom{4}{2}\binom{12}{3}4^3}{\binom{52}{5}}$ c. $\dfrac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{3}}{\binom{52}{5}}$ d. $\dfrac{\binom{52}{2}\binom{50}{3}}{\binom{52}{5}}$ e. None of these.

Solution. There are

$\binom{13}{1}$ ways to choose the face value for the pair,

$\binom{4}{2}$ ways to choose the suits for the pair,

$\binom{12}{3}$ ways to choose the face values for the other 3 cards,

4^3 ways to choose the suits for the other 3 cards,

$\binom{52}{5}$ ways to choose a hand (5 cards) in poker.

Problem 4. If $P(A|B) > P(A|C)$ then it follows that $P(B) > P(C)$.

a. True b. False ⓒ Cannot say based on information given.

Solution.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(A|C) = \frac{P(A \cap C)}{P(C)}.$$

If $P(A|B) > P(A|C)$ then to deduce that $P(B) > P(C)$ this must be true for all $P(A|C)$, in particular when $P(A|C) > 0$. Thus we can divide by it and get

$$\frac{P(A|B)}{P(A|C)} > 1 \Rightarrow \frac{P(A \cap B)P(C)}{P(B)P(A \cap C)} > 1.$$

But clearly we cannot reach any conclusions on just $P(B)$ and $P(C)$ from this because of the other terms involved. We can make $P(B) > P(C)$ or $P(B) < P(C)$ and still satisfy the inequality by adjusting the other terms.

Problem 5. Suppose that in a certain test to detect a disease it is known that the probability that a person tests positive given that the person has the disease is 0.99, and the probability that a person tests positive given that the person does not have the disease is 0.005 and the probability that a randomly selected person has the disease is 0.001. Find the probability that a person has the disease given that the person tests positive for the disease.

a. 0.015 ⓑ. 0.165 c. 0.325 d. None of these.

Solution.

$$P(D|+) = \frac{P(+|D)P(D)}{P(+)} = \frac{P(+|D)P(D)}{P(+|D)P(D) + P(+|\overline{D})P(\overline{D})}$$

$$= \frac{0.99(0.001)}{0.99(0.001) + 0.005(0.999)} = 0.165.$$

Problem 6. The joint *pdf* for (X, Y) is given by

$$f_{XY}(x, y) = \begin{cases} k, & 0 < x < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the constant k that makes this a valid *pdf*.

ⓐ 2 b. 1 c. $\sqrt{2}$ d. None of these.

Solution.

$$\int_0^1 \int_x^1 k dy dx = 1 \Rightarrow k = 2.$$

Problem 7. The joint *pdf* for (X, Y) is given by

$$f_{XY}(x, y) = \begin{cases} kxy, & 0 < x < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the constant k that makes this a valid *pdf*.

a. 1 b. $\sqrt{2}$ ⓒ 8 d. None of these.

Solution.

$$\int_0^1 \int_x^1 kxy dy dx = 1 \Rightarrow k = 8.$$

Problem 8. You get to appear on a game show. There are 3 doors (numbered 1 to 3)with prizes behind them. One door has a car as a grand prize, another door has a television and another door has a goat as a prize. You cannot see what is behind the doors. The game show host asks you to pick a door. Say you pick door number 2. Without revealing what is behind the door you chose, the host then opens door number 3 and reveals that it has the goat as a prize (note the host knows what is behind all the doors and always opens a door that does not have the grand prize behind it). At this point the host offers to let you exchange your door for door number 1. To maximize your chances of winning the car what should you do?

a. Keep door number 2. (b). Exchange your door for door number 1.

c. It does not matter since the probability of winning now is 0.5..

Solution. You should exchange your door for door number 1 so the answer is b. When you first chose a door there was a 1/3 probability of choosing the door with the car. At this point you would gladly exchange your door for the other 2 doors if you could since there is a 2/3 probability of one of those doors having the car. Now at least one of the doors you did not choose does not have a car so the fact that the host turns over a door that does not have the car does not change your 1/3 *a priori* probability of winning the car since the host knows what is behind each door and thus can always open a door that does not have the car. If you are have trouble seeing this try it at home with 3 cards. Pick an ace, 9 and a 2. Pretend the ace is the grand prize, the 9 is a good but not grand prize and the 2 is a joke prize. Turn the cards face down and pick one at random. Then peek at the other 2 cards and turn over one that is not an ace. If you think your odds are now 50-50 then you should be just as well off keeping your card everytime. With this stratgey in mind now turn over your card. Do this many times and see how often you win. If you do it a lot it should be about 1/3 of the time and not 1/2.

Problem 9. The continuous random variable X has *pdf*

$$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Find $P(X \leq 2)$.

a. e^{-4} ⓑ. $1 - \mathrm{e}^{-4}$ c. e^{-2} d. $1 - \mathrm{e}^{-2}$ e. None of these.

Solution.

$$P(X \leq 2) = \int_0^2 2e^{-2x} dx = 1 - e^{-4}.$$

Problem 10. Suppose $X \sim U(0,\ a)$ (uniform) and $Y \sim U(0,\ a)$. Let $Z = \max\{X, Y\}$. Find a so that $f_Z(z) = F_X(z)$.

a. $a = 1/2$ ⓑ. $\mathrm{a} = 2$ c. $\mathrm{a} = 4$ d. $\mathrm{a} = \sqrt{2}$ e. None of these.

Solution. You can assume here that X and Y are independent and z is between 0 and a since this is where X and Y are defined. Now

$$P(Z \leq z) = P(\max\{X, Y\} \leq z) = P(X \leq z, Y \leq z) = P(X \leq z)P(Y \leq z)$$

$$= F_X(z)F_Y(z) = \frac{z}{a}\frac{z}{a} = \frac{z^2}{a^2} \Rightarrow f_Z(z) = \frac{2z}{a^2}.$$

Set $f_Z(z) = F_X(z)$ to get

$$\frac{2z}{a^2} = \frac{z}{a} \Rightarrow a = 2.$$

Problem 11. The continuous random variable X has *pdf*

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Let $Y = e^{-X}$. Then the *pdf* of Y is which of the following.

a. exponential ⓑ. uniform c. logarithmic d. None of these.

Solution.

$$P(Y \leq y) = P(e^{-X} \leq y) = P(-X \leq \ln y) = P(X \geq lny) = 1 - F_X(-\ln y)$$

$$= 1 - (1 - e^{\ln y}) = y$$

so Y is uniform.

Problem 12. Suppose the random variable X has mean 2 and variance 6. Let $Y = X^2 + 1$. Find the mean of Y.

a. 5 b. 9 ⓒ 11 d. Cannot say based on information given. e. None of these.

Solution.

$$E(Y) = E(X^2) + 1 = Var(X) + [E(X)]^2 + 1 = 6 + 4 + 1 = 11.$$

Problem 13. Suppose the random variable X has mean 2 and variance 6. Let $Y = X^2 + 1$. Find the variance of Y.

a. 39 b. 22 c. 36 ⓓ Cannot say based on information given. e. None of these.

Solution.

$$Var(Y) = Var(X^2 + 1) = Var(X^2)$$

but we do not have enough information to compute the variance of X^2.

Problem 14. Suppose $X \sim N(2,\ 5)$,i.e., X is normally distributed with mean 2 and variance 5. Find the value of b that minimzes $E\left[(X-b)^2\right]$.

a. 5 ⓑ. 2 c. 4 d. 1 e. None of these.

Solution. We showed in class that the value of b that minimzes $E\left[(X-b)^2\right]$ is $b = E(X)$ which is $E(X) = \mu = 2$ in this case.

Problem 15. Suppose $X \sim N(2,\ 5)$. Find $E\left[X^4\right]$.

a. 75 b. 91 c. 25 ⓓ 211 e. None of these.

Solution. Let $Z = X - 2$. Then $Z \sim N(0,\ 5)$. Now $X = Z + 2$. Thus

$$X^4 = Z^4 + 8Z^3 + 24Z^2 + 32Z + 16$$

and

$$E\left(X^4\right) = E\left(Z^4 + 8Z^3 + 24Z^2 + 32Z + 16\right) = E\left(Z^4\right) + E\left(24Z^2\right) + 16$$

$$= 3(25) + 24(5) + 16 = 211.$$

Problem 16. Suppose the random variable X has mean 0 and variance 2. Let $Y = 2X + 1$. Find the correlation coefficient, r_{XY}.

ⓐ 1 b. 2 c. $\sqrt{2}$ d. Cannot say based on information given. e. None of these.

Solution. Since Y is of the form $Y = aX + b$ with $a > 0$ it follows that $r_{XY} = 1$.

Problem 17. Suppose $X \sim N(2,\ 4)$, and $Y \sim N(0,\ 6)$,. Let $Z = X + Y$. Find the variance of Z.

a. 24 b. 10 c. 6 ⓓ Cannot say based on information given. e. None of these.

Solution. We have

$$Var(Z) = Var(X+Y) \Rightarrow \sigma_Z^2 = \sigma_X^2 + 2r_{XY}\sigma_X\sigma_Y + \sigma_Y^2$$

but we do not know r_{XY}.

Problem 18. The two-dimensional continuous random variable (X, Y) has joint *pdf*

$$f_{XY}(x,y) = \begin{cases} x+y, & 0 \le x \le 1,\ 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a. Compute $f_{Y|X}(y|x)$.

Solution.

$$f_{Y|X}(y|x) = f(y|x) = \frac{f(x,y)}{f(x)} = \frac{x+y}{f(x)}.$$

Now

$$f(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 (x+y)dy = x + \frac{1}{2}.$$

So

$$f(y|x) = \begin{cases} \dfrac{x+y}{x+1/2}, & 0 \le x \le 1,\ 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

b. Find $E[Y|X = x]$.

Solution.

$$E[Y|X=x] = \int_{-\infty}^{\infty} yf(y|x)dy = \int_0^1 y\frac{x+y}{x+1/2}dy = \frac{3x+2}{6x+3}.$$

Problem 19 - **<u>Notice to Students:</u>** For the following Problem 19 you cannot decouple the $Y = Z_1 Z_2$ from the $X = Z_1 + Z_2$ as was done for a similar type problem given in class. You can still compute using expected values for the most part but you have to recognize that once a specific value of $X = x$ is given then Y is uniquely determined because of the coupling. Due to the fact that the example provided in class did not have this same type of dependency and the subtle nature of this problem, I decided to give you full credit for this problem.

Problem 19. Suppose Z_1 and Z_2 are (independent) bernoulli random variables with

$$E(Z_1) = E(Z_2) = p, \text{ and } Var(Z_1) = Var(Z_2) = pq.$$

Let $X = Z_1 + Z_2$ and $Y = Z_1 Z_2$. Find

a. $E(Y|X = x)$.

Solution. First of all note that since $E(Z_1) = E(Z_2) = p$ and $Var(Z_1) = Var(Z_2) = pq$ then

$$P(Z_1 = 1) = P(Z_2 = 1) = p, \quad P(Z_1 = 0) = P(Z_2 = 0) = 1 - p = q.$$

Now $X = 0$ if and only if $Z_1 = 0$ and $Z_2 = 0$, which implies $Y = 0$. $X = 1$ if and only $Z_1 = 1,\ Z_2 = 0$ or $Z_1 = 0,\ Z_2 = 1$, which implies $Y = 0$. $X = 2$ if and only if $Z_1 = 1$ and $Z_2 = 1$, which implies $Y = 1$. Thus,

$$E(Y|X = x) = \delta(x - 2).$$

b. $E[E(Y|X)]$.

Solution.

$$E[E(Y|X)] = E(Y|X = 0)P(X = 0) + E(Y|X = 1)P(X = 1)$$
$$+E(Y|X = 2)P(X = 2) = 0 + 0 + p^2 = p^2.$$

c. $Var[E(Y|X)]$.

Solution.

$$Var[E(Y|X)] = E\left[(E(Y|X))^2\right] - [E\,[E(Y|X)]]^2 = p^2 - p^4 = p^2(1-p^2).$$

d. $Var(Y|X = x)$.

Solution. Since Y is determined given a value of $X = x$ we have

$$Var(Y|X = x) = 0.$$

e. $E[Var(Y|X)]$.

Solution.

$$E[Var(Y|X)] = 0.$$

f. Best MSE predictor of Y given $X = x$.

Solution. The best MSE predictor is

$$E(Y|X = x) = \delta(x - 2).$$

g. Best linear MSE predictor of Y.

Solution. The best linear MSE predictor is

$$\hat{Y} = a_1 X + b_1$$

where,

$$a_1 = \frac{Cov(X, Y)}{Var(X)}, \quad b_1 = E(Y) - a_1 E(X).$$

So

$$a_1 = \frac{E(XY) - E(X)E(Y)}{Var(X)} = \frac{E[(Z_1 + Z_2)Z_1 Z_2] - E(Z_1 + Z_2)E(Z_1)E(Z_2)}{Var(Z_1 + Z_2)}$$

$$= \frac{p^2 + p^2 - 2pp^2}{2pq} = \frac{2p^2(1 - p)}{2pq} = p$$

and

$$b_1 = E(Z_1 Z_2) - a_1 E(Z_1 + Z_2) = p^2 - p(2p) = -p^2.$$

Thus,

$$\hat{Y} = pX - p^2.$$

Problem 20. Suppose we toss a fair coin 50 times. Let X denote the number of heads obtained.

a. Find $P(22 < X \leq 28)$ exactly.

Solution.

$$P(22 < X \leq 28) = \sum_{k=23}^{28} \binom{50}{k} (1/2)^{50} = 0.5989.$$

b. Find $P(22 < X \leq 28)$ using the normal approximation. Write your answer using

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du.$$

Solution.

$$\mu = np = 50(1/2) = 25,$$
$$\sigma^2 = npq = 50(1/2)(1/2) = 12.5 \Rightarrow \sigma = 3.5355.$$
$$P(22 < X \leq 28) = P\left(\frac{22-\mu}{\sigma} < \frac{X-\mu}{\sigma} \leq \frac{28-\mu}{\sigma}\right)$$
$$\approx P(-0.8485 < Z \leq 0.8485)$$

where $Z \sim N(0,\ 1)$. So

$$P(22 < X \leq 28) \approx \Phi(0.8485) - \Phi(-0.8485).$$

Note: $\Phi(0.8485) - \Phi(-0.8485) = 0.6038$.