EE 464 Exam 3 Solutions

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Inst: C.W. Walker

Problem	Points	Score
1	14	
2	14	
3	14	
4	16	
5	12	
6	12	
7	18	
Total	100	

Instructions and Information:

1) Print your name and location at the top of the page.

2) Make sure your exam has 7 problems and 18 numbered pages. Pages 8-9 are for extra workspace (reference them in the given problem if you use them – do not tear these out).

3) This is a closed book exam. A formula sheet is provided at the end of the exam. You may use a calculator. **You have 75 minutes to take this exam**.

4) Partial credit will be given but you must **show your work to receive any credit**.

5) **Circle or box your final answers**.

Problem 1. Consider the continuous random variable X with *pdf*

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Let $Y = 1/X$. Find the density function of Y.

Solution.

$$F_Y(y) = P(Y \le y) = P\left(\frac{1}{X} \le y\right) = P\left(X \ge \frac{1}{y}\right) = 1 - P\left(X < \frac{1}{y}\right)$$

$$= 1 - F_X\left(\frac{1}{y}\right).$$

$$\Rightarrow f_Y(y) = f_X\left(\frac{1}{y}\right)\frac{1}{y^2} = \frac{1}{y^2}\lambda e^{-\lambda/y}$$

so

$$f_Y(y) = \begin{cases} \frac{1}{y^2}\lambda e^{-\lambda/y}, & y > 0 \\ \\ 0, & \text{elsewhere.} \end{cases}$$

Problem 2. Suppose the two-dimensional ranndom variable (X, Y) has desnity

$$f(x, y) = \begin{cases} x + y, & 0 < x < 1,\ 0 < y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Compute $P(X + Y \ge 1/2)$.

Solution.

$$P(X+Y \ge 1/2) = 1 - P(X+Y < 1/2) = 1 - \int_0^{1/2}\int_0^{-x+1/2}(x+y)\,dy\,dx = \frac{23}{24}.$$

Problem 3.

a. Suppose X is exponentially distributed with parameter λ, i.e.,

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Derive the moment generating function for X.

Solution.

$$M_X(s) = E\left(e^{sX}\right) = \int_0^\infty e^{sx} \lambda e^{-\lambda x} dx = \frac{\lambda}{\lambda - s}, \ s < \lambda.$$

b. Suppose the random variable Y has moment generating function

$$M_Y(s) = (1-2s)^{-1}, \ s < \tfrac{1}{2}.$$

Find the mean and variance of Y.

Solution.

$$E(Y) = M_Y'(s)\Big|_{s=0} = -(1-2s)^{-2}(-2)\Big|_{s=0} = 2.$$

$$E(Y^2) = M_Y''(s)\Big|_{s=0} = 8(1-2s)^{-3}\Big|_{s=0} = 8.$$
$$\Rightarrow Var(Y) = 8 - 4 = 4.$$

Problem 4. Consider the random variable X with the Pareto density

$$f(x) = \begin{cases} \lambda x^{-\lambda-1}, & x > 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Let $Y = \ln(X)$ (the natural log). Find the density function of Y.

Solution.

$$F_Y(y) = P(Y \le y) = P(\ln(X) \le y) = P(X \le e^y) = F_X\left(e^y\right)$$

$$\Rightarrow f_Y(y) = f_X\left(e^y\right) \frac{d}{dy}\left(e^y\right) = \lambda \left(e^y\right)^{-\lambda-1} e^y$$

so

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y}, & y > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Problem 5. Recall that if X is normally distributed random variable then it has a moment generating function given by

$$M_X(s) = e^{s\mu + \sigma^2 s^2/2}.$$

Suppose that we have a standard normal random variable X (so the mean of X is 0 and the variance of X is 1).

Using the moment generating function compute $E\left[e^{4X}\right]$.

Solution.

$$M_X(s) = e^{s^2/2} = E\left[e^{sX}\right] \Rightarrow E\left[e^{4X}\right] = M_X(s)\Big|_{s=4} = e^8.$$

Problem 6. Suppose the random variable X has density

$$f_X(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Let $Y = X^2$. Find the expected value of Y.

Solution.

$$E(Y) = E(X^2) = \int_0^\infty 3x^2 e^{-3x} dx = \frac{2}{9}.$$

Problem 7. Suppose we have two random variables X and Y with respective densities

$$f_X(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

and

$$f_Y(y) = \begin{cases} y/2, & 0 < y < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Let $W = X/Y$. Find the density function for W.

Solution. Let

$$W = X/Y,\ Z = Y \Rightarrow X = ZW,\ Y = Z.$$

$$J = \begin{vmatrix} z & w \\ 0 & 1 \end{vmatrix}^{-1} = \frac{1}{z}.$$

$$f_{ZW}(z, w) = f_{XY}(wz, z)\frac{1}{|J|} = z f_X(wz) f_Y(z) = \frac{z^2}{2}e^{-wz}.$$

$$f_W(w) = \int_0^2 \frac{z^2}{2}e^{-wz}dz$$

we get

$$f_W(w) = \begin{cases} e^{-2w}\left[-\frac{2}{w} - \frac{2}{w^2} - \frac{1}{w^3}\right] + \frac{1}{w^3}, & w > 0 \\ 0, & \text{elsewhere.} \end{cases}$$