Name: _________________________ **Viewing Location: _____________________**

EE 464 Exam 3

April 23, 2003

Inst: C.W. Walker

Problem	Points	Score
1	14	
2	14	
3	14	
4	16	
5	12	
6	12	
7	18	
Total	100	

Instructions and Information:

1) Print your name and location at the top of the page.

2) Make sure your exam has 7 problems and 18 numbered pages. Pages 8-9 are for extra workspace (reference them in the given problem if you use them – do not tear these out).

3) This is a closed book exam. A formula sheet is provided at the end of the exam. You may use a calculator. **You have 75 minutes to take this exam**.

4) Partial credit will be given but you must **show your work to receive any credit**.

5) **Circle or box your final answers**.

Problem 1. Consider the continuous random variable X with *pdf*

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Let $Y = 1/X$. Find the density function of Y.

Problem 2. Suppose the two-dimensional ranndom variable (X, Y) has desnity

$$f(x,y) = \begin{cases} x+y, & 0 < x < 1,\ 0 < y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Compute $P(X + Y \geq 1/2)$.

Problem 3.

a. Suppose X is exponentially distributed with parameter λ, i.e.,

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Derive the moment generating function for X.

b. Suppose the random variable Y has moment generating function

$$M_Y(s) = (1 - 2s)^{-1}, \ s < \tfrac{1}{2}.$$

Find the mean and variance of Y.

Problem 4. Consider the random variable X with the Pareto density

$$f(x) = \begin{cases} \lambda x^{-\lambda-1}, & x > 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Let $Y = \ln(X)$ (the natural log). Find the density function of Y.

Problem 5. Recall that if X is normally distributed random variable then it has a moment generating function given by

$$M_X(s) = e^{s\mu + \sigma^2 s^2/2}.$$

Suppose that we have a standard normal random variable X (so the mean of X is 0 and the variance of X is 1).

Using the moment generating function compute $E\left[e^{4X}\right]$.

Problem 6. Suppose the random variable X has density

$$f_X(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Let $Y = X^2$. Find the expected value of Y.

Problem 7. Suppose we have two random variables X and Y with respective densities

$$f_X(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

and

$$f_Y(y) = \begin{cases} y/2, & 0 < y < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Let $W = X/Y$. Find the density function for W.

Extra workspace. If you use this space for work to be graded reference it from the given problem.

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Exam 3 Notes

9.1 Finding the Distribution of $g(X)$

9.1.1 Discrete Case

General Procedure:

1) First we consider the case where X is discrete and Y is discrete.

Let $x_{i1}, x_{i2}, \ldots$ represent the X-values having the property $g(x_{ij}) = y_i, \ \forall \ j$. Then

$$\begin{aligned} f_Y(y_i) = P(Y = y_i) &= P(X = x_{i1}) + P(X = x_{i2}) + \cdots \\ &= f_X(x_{i1}) + f_X(x_{i2}) + \cdots \end{aligned}$$

2) It may turn out that X is a continuous random variable while Y is discrete. If $\{Y = y_i\}$ is equivalent to an event, say A, in the range space of X, then

$$f_Y(y_i) = P(Y = y_i) = \int_A f_X(x)dx.$$

9.1.2 Continuous Case

Here X is a continuous random variable and g is a continuous function. So, $Y = g(X)$ is a continuous random variable. We seek the *pdf* of Y, i.e., $f_Y(y)$.

General Procedure:

i. Obtain $F_Y(y) = P(Y \leq y)$ by finding the event A (in the range space of X) which is equivalent to the event $\{Y \leq y\}$.

ii. Differentiate $F_Y(y)$ to get $f_Y(y)$.

iii. Determine those values in the range space of Y for which $f_Y(y) > 0$.

Theorem: Let X be a continuous random variable with *pdf* $f_X(x) > 0$ for $a < x < b$. Suppose that $y = g(x)$ is a strictly monotone (strictly increasing or strictly decreasing) function of x. Assume that g is differentiable (and hence continuous) for all x. Then $Y = g(X)$ has *pdf*

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

where x is expressed in terms of y, i.e., $x = g^{-1}(y)$. Hence.

$$f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\frac{dg^{-1}(y)}{dy}\right|.$$

If g is increasing then g is nonzero for those values of y satisfying $g(a) < y < g(b)$. If g is decreasing then g is nonzero for y satisfying $g(b) < y < g(a)$.

Theorem: Let X be a continuous random variable with *pdf* $f_X(x)$. Let $Y = X^2$. Then the *pdf* of Y is

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right].$$

Inverse Problem:

i. Given a random variable X with distribution function $F_X(x)$, find $g(x_0)$ so that $U = g(X)$ is uniform in (0, 1).

   Claim $g(x_0) = F_X(x_0)$ works.

ii. Given a random variable $U \sim U(0,1)$, find $g(u_0)$ so that $Y = g(U)$ has some desired distribution function $F_Y(y_0)$.

   Claim $g(u_0) = F_Y^{-1}(u_0)$ works.

iii. Given X with distribution function $F_X(x_0)$, find $g(x_0)$ so that $Y = g(X)$ has some desired distribution function $F_Y(y_0)$.

   Claim $g(x_0) = F_Y^{-1}(F_X(x_0))$ works.

9.2 Expectations

9.2.1 Discrete Case

Definition: The *expected value* of X is given by

$$E(X) = \sum_i x_i f(x_i)$$

whenever this sum exists.

Lemma: $E[g(X)] = \sum_i g(x_i) f_X(x_i)$ where $f_X(x_i) = P(X = x_i) = p_i$.

9.2.2 Continuous Case

Definition: The *expected value* of X is given by

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$$

whenever this integral exists.

Lemma: $E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$.

9.3 Variance

9.3.1 Discrete Case

Definition: The *variance* of X is given by

$$Var(X) = \sum_i (x_i - \mu)^2 f_X(x_i)$$

where $\mu = E(X)$. Thus, $Var(X) = E\left[(X - \mu)^2\right]$.

9.3.2 Continuous Case

Definition: The *variance* of X is given by

$$Var(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f_X(x) dx$$

where $\mu = E(X)$. Again, $Var(X) = E\left[(X - \mu)^2\right]$.

9.4 Examples and Additional Results

Theorem: Let X be binomially distributed with parameters n, p (write $X \sim B(n, p)$). Then $E(X) = np$.

Theorem: $Var(X) = E\left(X^2\right) - [E(X)]^2$.

Lemma: If X is discrete and takes values $1, 2, 3, \ldots$, then

$$E(X) = \sum_{n=0}^{\infty} P(X > n).$$

An Interpretation of Expectation:

$$\min_b E\left[(X-b)^2\right] = E\left[(X - E(X))^2\right].$$

9.5 Moments

Definitions: For $k = 1, 2, 3, \ldots$, the *kth moment* of X is

$$m_k = E\left[X^k\right]$$

and the *kth central moment* of X is

$$\mu_k = E\left[(X - E(X))^k\right].$$

Note: The 2nd central moment of X is the variance of X, $Var(X) = \sigma_X^2$. The *standard deviation* of X is $\sigma_X = \sqrt{\sigma_X^2}$.

Normal Case: Consider the mean-zero normal density

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}.$$

Claim: For $n \geq 1$,

$$E\left[X^n\right] = \begin{cases} 0, & n \text{ odd}, \\ 1 \cdot 3 \cdot 5 \cdots (n-1)\sigma^n, & n \text{ even}. \end{cases}$$

Theorem: (*Tchebycheff Inequality*). For any $\epsilon > 0$,

$$P\left(|X - \mu| \geq \epsilon\right) \leq \frac{\sigma^2}{\epsilon^2},$$

where $\mu = E(X)$ and $\sigma^2 = Var(X)$.

9.6 Moment Generating Function

Definition: Let X be a random variable. The *moment generating function* (*mgf*) of X is given by

$$M_X(s) = M(s) = E\left(e^{sX}\right).$$

For X discrete, the *mgf* of X is $M_X(s) = \sum_i e^{sx_i} P(X = x_i)$.

For X continuous, the *mgf* of X is $M_X(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x) dx$.

Theorem: $M_X^{(n)}(0) = E(X^n)$.

9.6.1 Examples

Binomial Case: Say X is binomially distributed with parameters n, p. We have

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \ k = 1, 2, 3, \ldots, n.$$

$$M'_X(0) = np = E(X).$$

$$Var(X) = E\left(X^2\right) - [E(X)]^2 = np[1 + (n-1)p] - (np)^2 = np(1-p).$$

9.7 Characteristic Functions

Definition: Let X be any random variable. The *characteristic function* (*cf*) of X is given by

$$\Phi_X(\omega) = E\left(e^{i\omega X}\right).$$

For X discrete, the *cf* of X is $\Phi_X(\omega) = \sum_k e^{i\omega x_k} P(X = x_k)$.

For X continuous, the *cf* of X is $\Phi_X(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f_X(x) dx$.

Note:

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_X(\omega) e^{-i\omega x} d\omega.$$

9.8 Special Moment Functions

Definition: If X is a random variable taking integer values, then its *moment function* is given by

$$\Gamma(z) = E\left(z^X\right) = \sum_i p_i z^i$$

where $p_i = P(X = i)$.

We get

$$\Gamma^{(k)} = E\left[X(X-1)\cdots(X-k+1)\right].$$

Special Case: If X is discrete taking values $0, 1, 2, \ldots$ the *probability generating function* of X is the function

$$G_X(s) = E\left(s^X\right), \; s \in \mathbf{R}, \quad \text{or } G_X(s) = \sum_{i=0}^{\infty} s^i P(X = i).$$

9.9 Applications of Characteristic Functions

If we can write $\int_{-\infty}^{\infty} e^{i\omega g(x)} f_X(x) dx$ as $\int_{-\infty}^{\infty} e^{i\omega y} h(y) dy$ then $f_Y(y) = h(y)$.

10.1 Joint Distribution and Density

Definition: The *joint distribution function*, $F_{XY}(x, y)$, of two random variables, X, Y, is defined by

$$F_{XY}(x, y) = P(X \leq x, Y \leq y).$$

Properties:

v.

$$\begin{aligned} P\left(x_1 < X \leq x_2, Y \leq y\right) &= P\left(X \leq x_2, Y \leq y\right) - P\left(X \leq x_1, Y \leq y\right) \\ &= F(x_2, y) - F(x_1, y). \end{aligned}$$

vi.

$$\begin{aligned} P\left(X \leq x, y_1 < Y \leq y_2\right) &= P\left(X \leq x, Y \leq y_2\right) - P\left(X \leq x, Y \leq y_1\right) \\ &= F(x, y_2) - F(x, y_1). \end{aligned}$$

vii.

$$P(x_1 < X \leq x_2, \; y_1 < Y \leq y_2) = F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1).$$

Definition: The *joint density function*, $f_{XY}(x, y)$, of two random variables, X, Y, is defined by

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y}.$$

Therefore,

$$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(u,v)dudv.$$

Definition: In the discrete case we have the *joint probabilities*

$$p_{ik} = P(X = x_i, Y = y_k).$$

Definitions: For two random variables X and Y, $F_X(x)$ is called the *marginal distribution* of X and $f_x(x)$ is called the *marginal density* of X. Similarly, for $F_Y(y)$ and $f_Y(y)$.

Claim:

$$F_X(x) = F(x,\infty),\ F_Y(y) = F(\infty, y),$$
$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy,\ f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx.$$

Definition: In the discrete case we have *marginal probabilities*, p_i and q_k, where $p_i = P(X = x_i)$ and $q_k = P(Y = y_k)$.

Here

$$p_i = \sum_k p_{ik}, \quad q_k = \sum_i p_{ik}.$$

Definition: We say (X,Y) is *uniformly distributed* over a region R in the Euclideam plane if

$$f(x,y) = \begin{cases} \dfrac{1}{\text{area of R}}, & \text{for } (x,y) \in R, \\ 0, & \text{elsewhere.} \end{cases}$$

10.2 Independence

Definition: Let (X,Y) be a two-dimensional discrete random variable. We say X and Y are *independent* if $p_{ik} = p_i p_k$, where $p_{ik} = P(X = x_i, Y = y_k)$, $p_i = P(X = x_i)$, $p_k = P(Y = y_k)$, i.e., $P(X = x_1, Y = y_k) = P(X = x_i)P(Y = y_k)\ \forall\ i,k$.

Definition: Let (X,Y) be a two-dimensional continuous random variable. We say X and Y are *independent* if $f(x,y) = f_X(x)f_Y(y)$, or equivalently $F(x,y) = F_X(x)F_Y(y)$, or equivalently $P(X \in S_X, Y \in S_Y) = P(X \in$

$S_X)P(Y \in S_Y)$, where S_X and S_Y are arbitrary measurable sets on the x-axis and y-axis, respectively.

Theorem: Let (X, Y) be a two-dimensional random variable. Let A and B be events whose occurrence (or nonoccurrence) depends only on X and Y, respectively. (That is, $A \subset R_X,\ B \subset R_Y$.) Then, if X and Y are independent random variables, $P(A \cap B) = P(A)P(B)$.

10.3 One Function of Two Random Variables

Given random variables X and Y and a function $g(x, y)$, we form the random variable

$$Z = g(X, Y).$$

We find the distribution function as

$$F_Z(z) = P(Z \leq z) = P(g(X, Y) \leq z) = P((x, y) \in D_z)$$

where, D_z is the region in the xy-plane consisting of the set

$$\{(x, y) : x \in R_X,\ y \in R_Y,\ g(x, y) \leq z\}.$$

So,

$$F_Z(z) = \int\int_{D_z} f(x, y)dxdy$$

where, $f(x, y)$ is the joint density of X and Y.

Definition: Two random variables X and Y are said to be *jointly normal* if

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1 - r^2}}$$

$$\cdot \exp\left[\frac{-1}{2(1 - r^2)}\left(\frac{(x - \mu_X)^2}{\sigma_X^2} - 2r\frac{(x - \mu_X)(y - \mu_Y)}{\sigma_X\sigma_Y} + \frac{(y - \mu_Y)^2}{\sigma_Y^2}\right)\right].$$

10.4 Two Functions of Two Random Variables

Here we have two random variables X and Y and form

$$Z = g(X, Y),\ W = h(X < Y).$$

$$F_{ZW}(z, w) = P(Z \le z, W \le w) = P(g(X, Y) \le z, h(X, Y) \le w).$$

<u>Theorem:</u> Let X and Y be two random variables and let $Z = g(X, Y)$ and $W = h(X, Y)$. Let $g(x, y) = z$ and $h(x, y) = w$ have solutions $(x_n, y_n), n = 1, 2, \ldots$. Then

$$f_{ZW}(z, w) = \frac{f_{XY}(x_1, y_1)}{|J(x_1, y_1)|} + \frac{f_{XY}(x_2, y_2)}{|J(x_2, y_2)|} + \cdots$$

where J is the Jacobian defined as

$$J(x, y) = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}^{-1}$$

<u>Theorem:</u> Let (X, Y) be a two-dimensional random variable and assume X and Y are independent. Let $W = XY$. Then

$$f_W(w) = \int_{-\infty}^{\infty} f_X(z) f_Y(w/z) \left| \frac{1}{z} \right| dz.$$

<u>Theorem:</u> Let (X, Y) be a two-dimensional random variable and assume X and Y are independent. Let $Z = X/Y$. Then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(wz) f_Y(w) \, |w| \, dw.$$

<u>Theorem:</u> Let (X, Y) be a two-dimensional random variable and assume X and Y are independent. Let $Z = X + Y$. Then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(w) f_Y(z - w) dw.$$

<u>Theorem:</u> Let X and Y be two independent random variables each of which may assume only nonnegative integral values. Let $p_k = P(X = k), k = 0, 1, 2, \ldots$. Let $q_r = P(Y = r), r = 0, 1, 2, \ldots$. Let $W = X + Y$ and let $w_i = P(W = i)$. Then

$$w_i = \sum_{k=0}^{i} p_k q_{i-k}, \; i = 0, 1, 2 \ldots.$$