EE 464 Exam 2 Solutions

March 10, 2003

Inst: C.W. Walker

Problem	Points	Score
1	12	
2	12	
3	12	
4	15	
5	16	
6	18	
7	15	
Total	100	

Instructions and Information:

1) Print your name and location at the top of the page.

2) Make sure your exam has 7 problems and 17 numbered pages. Pages 12-13 are for extra workspace (reference them in the given problem if you use them – do not tear these out). Do not write any work to be graded on the back of the pages.

3) This is a closed book exam. A formula sheet is provided at the end of the exam. You may use a calculator. **You have 75 minutes to take this exam**.

4) Partial credit will be given but you must **show your work to receive any credit**.

5) **Circle or box your final answers**.

Problem 1. Suppose we toss two fair dice. Let A be the event that the first die shows an odd number and let B be the event the second die shows an odd number. Let C be the event the sum of the two dice is odd. Show that events A, B, C are pairwise independent, i.e.,

$$P(A \cap B) = P(A)P(B),\ P(B \cap C) = P(B)P(C),\ P(A \cap C) = P(A)P(C)$$

but A, B, C are not independent.

Solution:

$$P(A) = P(B) = P(C) = \frac{18}{36} = \frac{1}{2}$$

$$P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{9}{36} = \frac{1}{4}$$

so

$$P(A \cap B) = P(A)P(B), \quad P(A \cap C) = P(A)P(C), \quad P(B \cap C) = P(B)P(C).$$

Hence events A, B, C are pairwise independent but

$$P(A \cap B \cap C) = P(\emptyset) = 0 \neq P(A)P(B)P(C) = \frac{1}{8}$$

thus A, B, C are not independent.

Problem 2. A coin is flipped and if the result is a head the random variable Y is formed by choosing a number uniformly in the interval [0, 1], i.e., $Y \sim U[0,\ 1]$. If the result is a tail then the random variable Y is formed by choosing a number uniformly in the interval [-1, 1], i.e., $Y \sim U[-1,\ 1]$.

Find $P\left(Y \leq \frac{2}{3}\right)$.

Solution:

$$P\left(Y \leq \frac{2}{3}\right) = P\left(Y \leq \frac{2}{3} \Big| H\right) P(H) + P\left(Y \leq \frac{2}{3} \Big| T\right) P(T)$$

$$= \frac{2}{3} \cdot \frac{1}{2} + \frac{5}{6} \cdot \frac{1}{2} = \frac{3}{4}.$$

Problem 3. A transmitter sends a signal over a channel. The signal s sent is either a +1 or a -1. During transmission the signal is corrupted by noise so that the receiver receives

$$Y = s + N$$

where N is a Gaussian random variable with mean zero ($\mu = 0$) and unit variance ($\sigma^2 = 1$). The receiver decides that a +1 was sent if $Y \geq 0$ and decides a -1 was sent if $Y < 0$.

a. Suppose that a +1 is sent by the transmitter. Find the probability that the receiver makes an error. Write you answer using the Q-function where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2} dt.$$

Solution: Note: The Q-function should have been defined as

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt$$

but no points were deducted if you did not notice the Q-function given had the incorrect exponent.

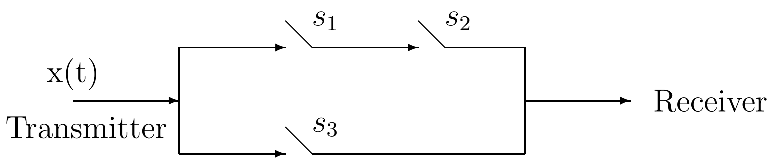
$$P(error) = P(Y \leq 0|s = +1) = P(s + N \leq 0|s = +1) = P(1 + N \leq 0)$$
$$= P(N \leq -1) = P(N \geq +1) = Q(1).$$

b. To guard against wrong decisions by the receiver the transmitter sends a +1 three times in a row. The receiver then makes three separate and independent decisions (one for each signal) and then uses majority logic (i.e., it looks at which decision, +1 or -1, occurred the most out of the three decisions) to make a final decision on which signal was sent. Find the probablity that receiver makes an error in this case.

Solution:

$$P(error) = \binom{3}{2} [Q(1)]^2 [1 - Q(1)] + [Q(1)]^3.$$

Problem 4. Consider the transmission of a signal as shown in the following diagram.



A signal is transmitted along two paths as shown. In the upper path there are two switches to pass through while in the lower path there is one switch to pass through. If a switch operates correctly then the signal passes through it. Switch s_1 operates independently of the other two switches and allows the signal to pass with probability $p_1 = 1/2$. It is known that switch s_2 operates correctly with probability $p_2 = 2/3$ if switch s_3 is operating correctly; if s_3 is not operating correctly then switch s_2 operates correctly with probability $p_2 = 1/3$. The probability that switch s_3 operates correctly is $p_3 = 1/2$. The signal transmission is successful if the signal $x(t)$ sent at the transmitter reaches the receiver along either or both paths.

Find the probability that the transmission is successful.

Solution:

$$P(success) = P(success|s_3 \text{ operates correctly})P(s_3 \text{ operates correctly})$$

$$+P(success|s_3 \text{ does not operate correctly})P(s_3 \text{ does not operate correctly}).$$

Now if s_3 operates correctly then $P(success) = 1$ since the signal goes thru the bottom path. If s_3 does not operate correctly then $P(success)$ is the probability of passing thru the top path since it cannot go thru the bottom path in this case. Thus,

$$P(success) = 1 \cdot \frac{1}{2} + \left(\frac{1}{2} \cdot \frac{1}{3}\right)\frac{1}{2} = \frac{7}{12}.$$

Problem 5. Let X be a geometric random variable, i.e.,

$$P(X = k) = (1-p)^{k-1}p, \ k = 1, 2, \ldots$$

Find the conditional distribution function $F_X(x|A)$ where A is the event

a. $A = \{X > s\}$.

Solution:

$$\begin{aligned} F_X(x|A) &= P(X \le x|A) = \frac{P(X \le x, \ A)}{P(A)} = \frac{P(X \le x, \ X > s)}{P(X > s)} \\ &= \frac{F_X(x) - F_X(s)}{1 - F_X(s)}. \end{aligned}$$

Now

$$P(A) = P(X > s) = \sum_{k=s+1}^{\infty} (1-p)^{k-1}p = (1-p)^s$$

so,

$$F_X(x) = P(X \le x) = 1 - P(X > x) = 1 - (1-p)^x$$

thus,

$$F_X(x|A) = \frac{-(1-p)^x + (1-p)^s}{(1-p)^s}$$

or

$$F_X(x|A) = \begin{cases} 1 - (1-p)^{x-s}, & x > s \\ 0, & \text{elsewhere.} \end{cases}$$

b. $A = \{X < s\}$.

Solution.

$$P(A) = P(X < s) = \sum_{k=1}^{s-1} (1-p)^{k-1}p = 1 - (1-p)^{s-1}.$$

Thus,

$$F_X(x|A) = \begin{cases} \dfrac{1-(1-p)^x}{1-(1-p)^{s-1}}, & x < s \\ 1, & \text{elsewhere.} \end{cases}$$

Problem 6. The continuous random variable X has *pdf*

$$f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

a. Find the distribution function $F_X(x)$.

Solution.

$$F_X(x) = P(X \leq x) = \int_0^x 3e^{-3u} du = 1 - e^{-3x}.$$

b. Find $P(X \leq 1)$.

Solution.

$$P(X \leq 1) = F_X(1) = 1 - e^{-3}.$$

c. Find $P(X \geq 2 | X \geq 1)$.

Solution.

$$P(X \geq 2 | X \geq 1) = \frac{P(X \geq 2, X \geq 1)}{P(X \geq 1)} = \frac{P(X \geq 2)}{P(X \geq 1)} = \frac{1 - F_X(2)}{1 - F_X(1)} = e^{-3}.$$

Problem 7. A man is saving up to buy a new car at a cost of N units of money. He starts with k units of money $(0 < k < N)$ and tries to win the remainder with the following gamble with his bank manager. He tosses a fair coin; if it turns up heads the bank manager pays him one unit of money, but if it comes up tails then he pays the manager one unit of money. He keeps tossing the coin and playing this game until either he has won enough units of money to buy the car or he loses his k units of money (goes bankrupt).

Let A_k denote the event that he is eventually bankrupt after his initial capital was k units. Let $p_k = P(A_k)$.

a. Show $p_k = \frac{1}{2}(p_{k+1} + p_{k-1})$ if $0 < k < N$.

   Solution. Let B be the event of getting heads on the first toss of the coin. Then,

$$\begin{aligned} P(A_k) &= P(A_k|B)\,P(B) + P\left(A_k|\bar{B}\right)P(\bar{B}) \\ &= P(A_{k+1})\,\frac{1}{2} + P(A_{k-1})\,\frac{1}{2} \end{aligned}$$

   so

$$p_k = \frac{1}{2}(p_{k+1} + p_{k-1}).$$

b. The result in part (a) is a linear difference equation subject to the boundary conditions $p_0 = 1,\ p_N = 0$. Solve this difference equation for p_k. Hint: If you have not solved this type of equation analytically before then you can instead proceed directly as follows: First let $b_k = p_k - p_{k-1}$. Show $b_k = b_{k-1}$ and thus $b_k = b_1$ for all k. Continue from here.

Solution. Define

$$b_k = p_k - p_{k-1}.$$

Then,

$$b_{k-1} = p_{k-1} - p_{k-2}.$$

So,

$$b_k + b_{k-1} = p_k - p_{k-2}.$$

Also,

$$p_k = \frac{1}{2}\left(p_{k+1} + p_{k-1}\right) \Rightarrow p_{k-1} = \frac{1}{2}\left(p_k + p_{k-2}\right)$$

so

$$p_{k-1} - p_{k-2} = \frac{1}{2}\left(p_k - p_{k-2}\right).$$

Therefore,

$$b_{k-1} = \frac{1}{2}\left(b_k + b_{k-1}\right)$$

and thus

$$b_{k-1} = b_k \Rightarrow b_k = b_1.$$

Hence,

$$b_1 = p_1 - p_0 = p_1 - 1$$
$$b_2 = p_2 - p_1 = p_1 - 1 \Rightarrow p_2 = 2p_1 - 1$$
$$b_3 = p_3 - p_2 = p_3 - 2p_1 + 1 = p_1 - 1 \Rightarrow p_3 = 3p_1 - 2.$$

Similarly,

$$p_k = kp_1 - (k-1)$$

and

$$p_N = Np_1 - (N-1) \Rightarrow 0 = Np_1 - (N-1) \Rightarrow p_1 = \frac{N-1}{N}.$$

Thus

$$p_k = k\frac{N-1}{N} - (k-1) \Rightarrow p_k = 1 - \frac{k}{N}.$$