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**EE 464 Exam 2**

**March 10, 2003**

**Inst: C.W. Walker**

<b>Problem</b>	<b>Points</b>	<b>Score</b>
<b>1</b>	<b>12</b>	
<b>2</b>	<b>12</b>	
<b>3</b>	<b>12</b>	
<b>4</b>	<b>15</b>	
<b>5</b>	<b>16</b>	
<b>6</b>	<b>18</b>	
<b>7</b>	<b>15</b>	
<b>Total</b>	<b>100</b>	

**Instructions and Information:**

- 1) Print your name and location at the top of the page.
- 2) Make sure your exam has 7 problems and 17 numbered pages. Pages 12-13 are for extra workspace (reference them in the given problem if you use them – do not tear these out). Do not write any work to be graded on the back of the pages.
- 3) This is a closed book exam. A formula sheet is provided at the end of the exam. You may use a calculator. **You have 75 minutes to take this exam.**
- 4) Partial credit will be given but you must **show your work to receive any credit.**
- 5) **Circle or box your final answers.**

**Problem 1.** Suppose we toss two fair dice. Let  $A$  be the event that the first die shows an odd number and let  $B$  be the event the second die shows an odd number. Let  $C$  be the event the sum of the two dice is odd. Show that events  $A, B, C$  are pairwise independent, i.e.,

$$P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(A \cap C) = P(A)P(C)$$

but  $A, B, C$  are not independent.

**Problem 2.** A coin is flipped and if the result is a head the random variable  $Y$  is formed by choosing a number uniformly in the interval  $[0, 1]$ , i.e.,  $Y \sim U[0, 1]$ . If the result is a tail then the random variable  $Y$  is formed by choosing a number uniformly in the interval  $[-1, 1]$ , i.e.,  $Y \sim U[-1, 1]$ .

Find  $P\left(Y \leq \frac{2}{3}\right)$ .

**Problem 3.** A transmitter sends a signal over a channel. The signal  $s$  sent is either a +1 or a -1. During transmission the signal is corrupted by noise so that the receiver receives

$$Y = s + N$$

where  $N$  is a Gaussian random variable with mean zero ( $\mu = 0$ ) and unit variance ( $\sigma^2 = 1$ ). The receiver decides that a +1 was sent if  $Y \geq 0$  and decides a -1 was sent if  $Y < 0$ .

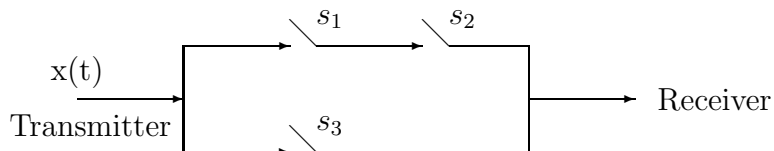
- a. Suppose that a +1 is sent by the transmitter. Find the probability that the receiver makes an error. Write your answer using the  $Q$ -function where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2} dt.$$

- b. To guard against wrong decisions by the receiver the transmitter sends a +1 three times in a row. The receiver then makes three separate and independent decisions (one for each signal) and then uses majority logic (i.e., it looks at which decision, +1 or -1, occurred the most out of the three decisions) to make a final decision on which signal was sent. Find the probability that the receiver makes an error in this case.

Extra workspace for Problem 3.

**Problem 4.** Consider the transmission of a signal as shown in the following diagram.



A signal is transmitted along two paths as shown. In the upper path there are two switches to pass through while in the lower path there is one switch to pass through. If a switch operates correctly then the signal passes through it. Switch  $s_1$  operates independently of the other two switches and allows the signal to pass with probability  $p_1 = 1/2$ . It is known that switch  $s_2$  operates correctly with probability  $p_2 = 2/3$  if switch  $s_3$  is operating correctly; if  $s_3$  is not operating correctly then switch  $s_2$  operates correctly with probability  $p_2 = 1/3$ . The probability that switch  $s_3$  operates correctly is  $p_3 = 1/2$ . The signal transmission is successful if the signal  $x(t)$  sent at the transmitter reaches the receiver along either or both paths.

Find the probability that the transmission is successful.

Extra workspace for Problem 4.

**Problem 5.** Let  $X$  be a geometric random variable, i.e.,

$$P(X = k) = (1 - p)^{k-1}p, \quad k = 1, 2, \dots$$

Find the conditional distribution function  $F_X(x|A)$  where  $A$  is the event

- a.  $A = \{X > s\}$ .
- b.  $A = \{X < s\}$ .



Extra workspace for Problem 5.

**Problem 6.** The continuous random variable  $X$  has *pdf*

$$f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

- a. Find the distribution function  $F_X(x)$ .
- b. Find  $P(X \leq 1)$ .
- c. Find  $P(X \geq 2|X \geq 1)$ .

**Problem 7.** A man is saving up to buy a new car at a cost of  $N$  units of money. He starts with  $k$  units of money ( $0 < k < N$ ) and tries to win the remainder with the following gamble with his bank manager. He tosses a fair coin; if it turns up heads the bank manager pays him one unit of money, but if it comes up tails then he pays the manager one unit of money. He keeps tossing the coin and playing this game until either he has won enough units of money to buy the car or he loses his  $k$  units of money (goes bankrupt).

Let  $A_k$  denote the event that he is eventually bankrupt after his initial capital was  $k$  units. Let  $p_k = P(A_k)$ .

- a. Show  $p_k = \frac{1}{2}(p_{k+1} + p_{k-1})$  if  $0 < k < N$ .
- b. The result in part (a) is a linear difference equation subject to the boundary conditions  $p_0 = 1$ ,  $p_N = 0$ . Solve this difference equation for  $p_k$ . Hint: If you have not solved this type of equation analytically before then you can instead proceed directly as follows: First let  $b_k = p_k - p_{k-1}$ . Show  $b_k = b_{k-1}$  and thus  $b_k = b_1$  for all  $k$ . Continue from here.

Extra workspace for Problem 7.

**Extra workspace.** If you use this space for work to be graded reference it from the given problem.

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## Exam 2 Notes

**Definition:** Events  $A_1, \dots, A_n$  are *independent* if for any  $k$ -tuple ( $1 \leq k \leq n$ ) we have

$$P(A_{i_1} \cap \dots \cap A_{i_k}) = P(A_{i_1}) \cdots P(A_{i_k})$$

where

$$i_j \in \{1, 2, \dots, n\} \text{ and } i_m \neq i_s, m \neq s.$$

We can view several experiments as one combined experiment.

**Definition:** The *cartesian product*  $\Omega_1 \times \Omega_2 \times \dots \times \Omega_n$  is the set of all  $n$ -tuples  $\{\omega_1 \omega_2 \cdots \omega_n\}$  such that  $\omega_k \in \Omega_k$ ,  $k = 1, 2, \dots, n$ .

**Definition:** For  $(\Omega, F, P)$ , a probability space, a *random variable* is a function  $X : \Omega \rightarrow \mathbf{R}$  (for each outcome  $\omega \in \Omega$  we have a number  $X(\omega) \in \mathbf{R}$ ) with the property that

$$\{\omega : X(\omega) \leq x\} \in F \quad \forall x \in \mathbf{R}.$$

**Definition:** The *distribution function* of a random variable  $X$  is the function

$$F_X = F : \mathbf{R} \rightarrow [0, 1]$$

given by

$$F(x) = P(\{\omega : X(\omega) \leq x\}) \quad \forall x \in \mathbf{R}.$$

**Notation:**  $\{\omega : X(\omega) \leq x\}$  is abbreviated  $\{X \leq x\}$ . So we write

$$F(x) = P(X \leq x) \text{ or } F_X(x) = P(X \leq x).$$

**Definition:** A random variable  $X$  is *discrete* if it takes values in a finite or countably infinite set  $\{x_1, x_2, \dots\}$ .

**Definition:** A random variable  $X$  is *continuous* if there exists a function

$$f : \mathbf{R} \rightarrow [0, \infty)$$

such that

$$F_X(x) = \int_{-\infty}^x f(u) du.$$

So  $F_X$  continuous implies  $P(X = x) = 0 \forall x$  and

$$P(a < X \leq b) = \int_a^b f(u)du.$$

We may have a combination of continuous and discrete distributions. These are called mixtures.

**Definition:** If  $X$  is continuous then

$$f_X(x) = \frac{dF_X(x)}{dx}$$

is called the *probability density function* of  $X$ . We may write  $f(x) = f_X(x)$  when dealing with just one random variable.

**Definition:** If  $X$  is discrete and  $P(X = x_i) = p_i$  then

$$f_X(x) = \sum_i p_i \delta(x - x_i)$$

is called the *probability mass function*.

Here,  $f(x_i) = p_i$  and

$$\delta(u) = \begin{cases} 1, & u = 0 \\ 0, & \text{elsewhere.} \end{cases}$$

In the continuous case

$$F_X(x) = \int_{-\infty}^x f_X(u)du$$

and in the discrete case

$$F_X(x) = \sum_{u:u \leq x} f_X(u).$$

**Normal or Gaussian:**  $X \sim N(\mu, \sigma^2)$ . It has the form

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad x \in \mathbf{R}.$$

The corresponding distribution function is

$$F(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-\mu)^2/2\sigma^2} dy.$$



**Uniform:**  $X \sim U(x_1, x_2)$ .. It has density

$$f(x) = \begin{cases} \frac{1}{x_2 - x_1}, & x_1 \leq x < x_2 \\ 0, & \text{elsewhere.} \end{cases}$$

**Bernoulli:**  $X$  is Bernoulli if it can take on only two values.

**Binomial:**  $X \sim B(n, p)$ ..  $X$  is binomial of order  $n$  if

$$P(X = k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, 2, \dots, n$$

where  $q = 1 - p$ . The mean of  $X$  is  $np$ . We can think of the binomial distribution in terms of a sum of Bernoulli trials.

**Geometric:**  $X$  is geometric if

$$P(X = k) = (1 - p)^{k-1} p.$$

Here  $X$  can be thought of as the number of trials needed before some event  $A$  occurs for the first time.

**Poisson:**  $X$  is Poisson if

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \dots, n$$

for some  $\lambda > 0$ .

**Definition:** The *conditional distribution*  $F(x|M)$  of the random variable  $X$ , given the event  $M$  occurs, is defined as

$$F(x|M) = P(X \leq x|M) = \frac{P(X \leq x, M)}{P(M)}$$

where  $P(M) \neq 0$ .

**Definition:** The *conditional density*  $f(x|M)$  is the derivative of  $F(x|M)$  with respect to  $x$ , i.e.,

$$f(x|M) = \frac{dF(x|M)}{dx} = \lim_{\Delta x \rightarrow 0} \frac{P(x < X \leq x + \Delta x|M)}{\Delta x}.$$

### Special Cases

i. Let  $M$  be the event  $\{X \leq a\}$  where  $P(X \leq a) \neq 0$ . Then,

$$f(x|X \leq a) = \begin{cases} 0, & x \geq a \\ \frac{f(x)}{F(a)}, & x < a. \end{cases}$$

ii. Let  $M$  be the event  $\{b < X \leq a\}$  where  $F(a) \neq F(b)$ . Then,

$$f(x|b < X \leq a) = \begin{cases} \frac{f(x)}{F(a) - F(b)}, & b \leq x < a \\ 0, & \text{else.} \end{cases}$$

Let  $A_1, \dots, A_n$  be a partition of  $\Omega$ . Let  $B = \{X \leq x\}$ . Then

$$P(A|X \leq x) = \frac{F(x|A)}{F(x)}P(A).$$

$$P(A|x_1 < X \leq x_2) = \frac{F(x_2|A) - F(x_1|A)}{F(x_2) - F(x_1)}P(A).$$

$$P(A) = \int_{-\infty}^{\infty} P(A|X = x)f(x)dx.$$

The above expression is the continuous version of the total probability theorem.

$$f(x|A) = \frac{P(A|X = x)f(x)}{\int_{-\infty}^{\infty} P(A|X = x)f(x)dx}.$$

The above expression is the continuous version of Bayes' theorem.

### A Sum Formula:

$$\sum_{k=M}^N a^k = \frac{a^M - a^{(N+1)}}{1 - a}, \quad a \neq 1.$$