EE 564

Midterm Solutions

March 10, 2014

Work all 10 problems.

Problem 1. [6 pts]. Suppose the normal random variable X has mean 1 and variance 3. Let $Y = (X-1)^2$. Find the mean and variance of Y.

Solution: Let $W = X - 1$. Then $W \sim N(0,3)$. $Y = W^2$.
$E[Y] = E[W^2] = 3$, $E[Y^2] = E[W^4] = 3 \times 3^2 = 27$, so $Var[Y] = 27 - 3^2 = 18$.

Problem 2. [12 pts]. Pulse Coded Modulation (PCM) is to be used to encode a signal. The signal ranges between the values -4 and +4. There are 3 bits or 8 levels (hence 8 code numbers) available. The levels assigned have symmetry like we demonstrated in class. The first three sample values obtained (before quantization) are 2.8, 0.2, and -2.4, respectively.

a. Find the quantized values for the three sample values.

   Solution: $2.8 \rightarrow 2.5$, $0.2 \rightarrow 0.5$, $-2.4 \rightarrow -2.5$.

b. Find the corresponding code numbers for the quantized values.

   Solution. $2.5 \rightarrow 6$, $0.5 \rightarrow 4$, $-2.5 \rightarrow 1$.

c. Find the corresponding PCM sequences for the code numbers.

   Solution: $6 = 110$, $4 = 100$, $1 = 001$.

d. Draw the corresponding PCM waveform for the PCM sequence using NRZ-L logic.

   Solution. See below.

Problem 3. [5 pts]. What favorably characterisitc does Manchester coding of waveforms offer the communication system designer?

Solution. It guarantees signal level transitions on each bit which can aid synchronization.

Problem 4. [10 pts]. Suppose that a narrowband (bandpass) signal is given by

$$s(t) = A_c \sin[2\pi f_c t + \theta(t))]$$

where

$$\theta(t) = \beta \sin(2\pi f_m t).$$

Here f_c is much greater than f_m. This is referred to as frequency modulation (FM). Find $u(t)$, the equivalent low pass representation of this signal.

Solution:

$$\begin{aligned} s(t) &= A_c \sin[2\pi f_c t + \beta \sin(2\pi f_m t)] \\ &= A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t) - \pi/2] \end{aligned}$$

so from the development given in the class notes we have

$$u(t) = A_c \exp[\beta \sin(2\pi f_m t) - \pi/2].$$

Problem 5. [12 pts]. Recall that for a given a real-valued function $x(t)$, its Hilbert transform, denoted $\hat{x}(t)$, is defined by

$$\hat{x}(t) = x(t) * h(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau$$

where

$$h(t) = \frac{1}{\pi t}$$

with corresponding frequency response

$$H(f) = \begin{cases} -j, & f > 0 \\ 0, & f = 0 \\ j, & f < 0. \end{cases}$$

Show that a signal $s(t)$ and its Hilbert transform are orthogonal, i.e., show

$$\int_{-\infty}^{\infty} s(t)\hat{s}(t)dt = 0.$$

Solution: Note that

$$F\left(s(\tau) \star \hat{s}(\tau)\right) = S^*(f)\hat{S}(f)$$

where $\star$ denotes correlation and F denotes the Fourier transform. Hence,

$$s(\tau) \star \hat{s}(\tau) = F^{-1}\left(S^*(f)\hat{S}(f)\right).$$

Therefore,

$$\int_{-\infty}^{\infty} s(t)\hat{s}(t-\tau)dt = \int_{-\infty}^{\infty} S^*(f)\hat{S}(f)e^{j2\pi f\tau}df.$$

Evaluating this last expression at $\tau = 0$ gives

$$\int_{-\infty}^{\infty} s(t)\hat{s}(t)dt = \int_{-\infty}^{\infty} S^*(f)\hat{S}(f)df.$$

Thus,

$$\begin{aligned} \int_{-\infty}^{\infty} s(t)\hat{s}(t)dt &= \int_{-\infty}^{\infty} S^*(f)\hat{S}(f)df \\ &= \int_{-\infty}^{\infty} S^*(f)\left[-j\text{sgn}(f)S(f)\right]df \\ &= -j\int_{-\infty}^{\infty} \text{sgn}(f)|S(f)|^2 df \\ &= 0 \end{aligned}$$

since the function inside the integral on the right hand side of the above expression is odd.

Problem 6. [15 pts]. Consider an ideal bandpass filter

$$H_{BP}(f) = \begin{cases} e^{-j2\pi(f-f_c)t_0}, & f_c - B \le f \le f_c + B \\ e^{-j2\pi(f+f_c)t_0}, & -f_c - B \le f \le -f_c + B. \end{cases}$$

Let $s(t) = Ar_T(t)\cos(2\pi f_c t)$ where

$$r_T(t) = \begin{cases} 1, & -T/2 \le t \le T/2 \\ 0, & \text{elsewhere.} \end{cases}$$

Determine the response of the bandpass system to the input $s(t)$. Your answer may utilize the sine integral where the sine integral is defined by

$$\text{Si}(u) = \int_0^u \frac{\sin\lambda}{\lambda}.$$

You may assume that $f_c T \gg 1$ so that $s(t)$ may be considered narrowband.

Solution: We can use the low-pass equivalent filter

$$H_{LP}(f) = \begin{cases} e^{-j2\pi f t_0}, & -B \le f \le B \\ 0, & \text{elsewhere.} \end{cases}$$

We find

$$h_{LP}(t) = 2B\text{sinc}[2B(t - t_0)]$$

and

$$s_{LP}(t) = Ar_T(t)$$

and thus

$$\begin{aligned} y_{LP}(t) &= s_{LP}(t) * h_{LP}(t) \\ &= 2AB\int_{-T/2}^{T/2} \frac{\sin[2\pi B(t - t_0 - \tau)]}{2\pi B(t - t_0 - \tau)} d\tau. \end{aligned}$$

Let $\lambda = 2\pi B(t - t_0 - \tau)$. Then,

$$\begin{aligned} y_{LP}(t) &= \frac{A}{\pi}\int_{-2\pi B(t-t_0-T/2)}^{2\pi B(t-t_0+T/2)} \frac{\sin\lambda}{\lambda} d\lambda \\ &= \frac{A}{\pi}\int_0^{2\pi B(t-t_0+T/2)} \frac{\sin\lambda}{\lambda} d\lambda - \frac{A}{\pi}\int_0^{2\pi B(t-t_0-T/2)} \frac{\sin\lambda}{\lambda} d\lambda \\ &= \frac{A}{\pi}[\text{Si}(2\pi B(t - t_0 + T/2)) - \text{Si}(2\pi B(t - t_0 - T/2))]. \end{aligned}$$

Hence,

$$y(t) = \frac{A}{\pi} Re[[\mathrm{Si}(2\pi B(t - t_0 + T/2)) - \mathrm{Si}(2\pi B(t - t_0 - T/2))]e^{j2\pi f_c t}].$$

Problem 7. [10 pts]. Suppose we wish to test the performance of a communication system. To do this we transmit a known binary sequence (known to both the sender and receiver). At the output of the receiver the user compares the 0 and 1 decisions made by the receiver and compares them to the known sequence. The outputs are independent of each other. If the true probability of bit error is $P_b = 10^{-3}$ find the probability that more than 120 errors will be observed in 100,000 bits.

Solution: Let X denote the number of errors observed in 100,000 bits. Then, X has a binomial distribution and

$$\begin{aligned} P(X > 120) &= 1 - P(X \leq 120) \\ &= 1 - \sum_{k=0}^{120} \binom{100,000}{k} P_b^k (1 - P_b)^{100,000-k} = 0.0226. \end{aligned}$$

Problem 8. [15 pts]. A BPSK signal is given by

$$s(t) = \begin{cases} A\cos(2\pi f t + \phi), & 0 \leq t \leq T \\ 0, & \text{elsewhere,} \end{cases}$$

where $\phi = 0$ if a 1 is sent and $\phi = \pi$ if a 0 is sent and $A > 0$. One way to recover the information bit as discussed in class is to multiply $s(t)$ by $\cos(2\pi f t)$ and integrate the result from 0 to T and then decide upon 1 if the output is positive and decide upon 0 if the output is negative. Let y represent the output of the integration. Suppose the signal is transmitted at a bit rate of 1 Mbit/s, $f = 4$ Hz, $\phi = 0$ and $A = 10$ mV.

a. [5 pts]. Compute the energy in $s(t)$.

Solution:

$$E = \frac{A^2 T}{2} = 5 \times 10^{-11} \text{ J}.$$

b. [5 pts]. Compute y using the algorithm logic provided above.

Solution:

$$y = \frac{AT}{2} = 5 \times 10^{-9}.$$

c. [5 pts]. Now suppose we add the noise process, $n(t)$ to $s(t)$ to form the received signal

$$r(t) = s(t) + n(t)$$

where $n(t)$ is additive white Gaussian noise (AWGN) with single sided power spectral density N_0, where $N_0 = 10^{-11}$ W/Hz. The signal power and energy are normalized relative to a $R = 1\ \Omega$ load. Using an optimal demodulator what is the probability of making a bit decision error in this case? Note that $W = V^2/R$.

Solution:

$$\begin{aligned} P_b &= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \\ &= Q(3.16) \\ &\Rightarrow P_b = 8 \times 10^{-4} \end{aligned}$$

where we used $E_b = \frac{A^2T}{2}$.

Problem 9. [5 pts]. Suppose that a certain signal, $v(t)$, has an autocorrelation function given by

$$\Phi_{vv}(\tau) = \begin{cases} \frac{A^2}{2}(T - |\tau|), & 0 \leq |\tau| < T \\ 0, & \text{elsewhere.} \end{cases}$$

What is the energy in the signal?

Solution:

$$E = \Phi_{vv}(0) = \frac{A^2 T}{2}.$$

Problem 10. [10 pts]. Suppose we wish to compute the probabilty of bit error using a simulation. We run the simulation for n trials and count X errors. Our estimate for the probability of bit error, P_b, is

$$\hat{P}_b = \frac{X}{n}.$$

Suppose in your simulation you count 150 errors out of 100,000 trials. Compute your estimate for the probability of bit error and construct a 95% CI about it (you may utilize the normal approximation in this problem).

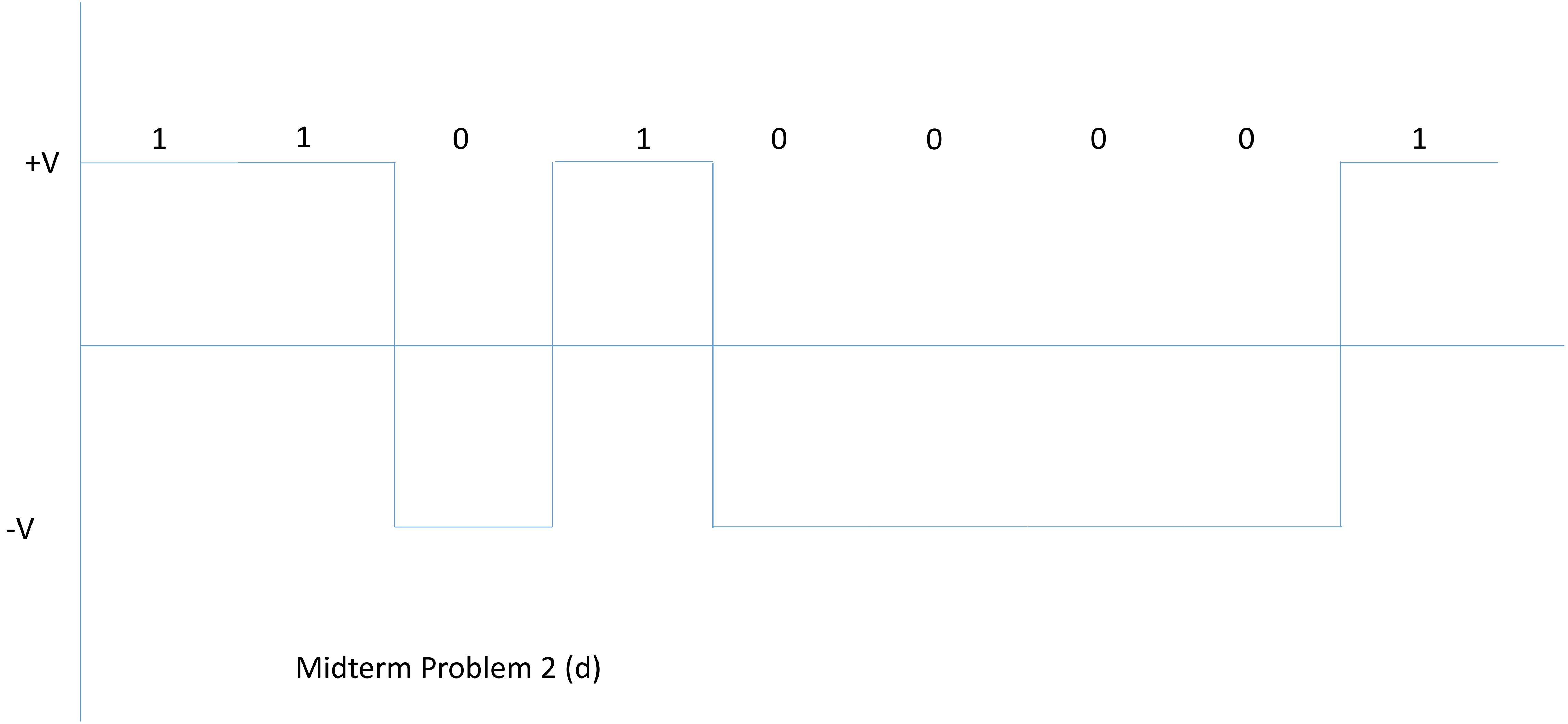
Solution:

$$\hat{p}_b = \frac{150}{100,000} = 0.0015.$$

The margin of error is

$$E = 1.96\sqrt{\frac{\hat{p}_b(1-\hat{p}_b)}{n}} = 0.00023987.$$

Hence, the 95% CI is $\hat{p}_b \pm E = (0.00126, 0.00174)$.



Midterm Problem 2 (d)