EE 564

Homework 8 Solutions

Problem 1. In class we had a model of a PLL and said the closed-loop transfer function was

$$H(s) = \frac{KG(s)/s}{1 + KG(s)/s}$$

where G(s) was the transfer function of the loop filter inside the PLL. Derive the above expression for $H(s)$ given our model.

Solution: We have using $\sin(\hat{\phi} - \phi) \approx \hat{\phi} - \phi$ for small $(\hat{\phi} - \phi)$

$$(\hat{\phi}(s) - \phi(s))G(s) \cdot \frac{1}{2}\frac{K}{s} = \hat{\phi}(s)$$

so

$$\frac{1}{2}\frac{KG(s)}{s}\phi(s) = \left(1 + \frac{1}{2}\frac{KG(s)}{s}\right)\hat{\phi}(s)$$

and thus

$$\begin{aligned} H(s) &= \frac{\hat{\phi}(s)}{\phi(s)} \\ &= \frac{KG(s)/s}{1 + KG(s)/s}. \end{aligned}$$

Note that the factor of 1/2 was absorbed into the gain K.

Problem 2. In class we indicated that $H(s)$ for the PLL mentioned in Problem 1 could be written as

$$H(s) = \frac{(2\zeta\omega_n - \omega_n^2/K)s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

The (one-sided) noise-equivalent bandwidth of the loop is defined as

$$B_{eq} = \frac{1}{G}\int_0^\infty |H(f)|^2 df$$

where $G = \max |H(f)|^2$. Show that

$$B_{eq} = \frac{1 + (\tau_2 \omega_n)^2}{8\zeta/\omega_n}$$

where we have utilized

$$G(s) = \frac{1 + \tau_2 s}{1 + \tau_1 s}$$

and

$$\omega_n = \sqrt{K/\tau_1}, \quad \zeta = \omega_n(\tau_2 + 1/K)/2.$$

Solution: We have

$$\begin{aligned} H(f) &= H(s)|_{s=2\pi jf} = \frac{(2\zeta\omega_n - \omega_n^2/K)2\pi jf + \omega_n^2}{(2\pi jf)^2 + 2\zeta\omega_n 2\pi jf + \omega_n^2} \\ &= \frac{(2\zeta\omega_n - \omega_n^2/K)2\pi jf + \omega_n^2}{-4\pi^2 f^2 + 4\pi\zeta\omega_n jf + \omega_n^2} \end{aligned}$$

so

$$|H(f)|^2 = \frac{(2\zeta\omega_n - \omega_n^2/K)2\pi jf + \omega_n^2}{-4\pi^2 f^2 + 4\pi\zeta\omega_n jf + \omega_n^2} \cdot \frac{-(2\zeta\omega_n - \omega_n^2/K)2\pi jf + \omega_n^2}{-4\pi^2 f^2 - 4\pi\zeta\omega_n jf + \omega_n^2}.$$

Let

$$\begin{aligned} a &= -(2\zeta\omega_n - \omega_n^2/K)j/2\pi \\ b &= -\omega_n^2/4\pi^2 \\ c &= -\zeta\omega_n j/\pi. \end{aligned}$$

Then,

$$|H(f)|^2 = \frac{af + b}{f^2 + cf + b} \cdot \frac{-af + b}{f^2 - cf + b}.$$

Define

$$f_1 = \frac{-c + \sqrt{c^2 - 4b}}{2}$$

$$f_2 = \frac{-c - \sqrt{c^2 - 4b}}{2}$$

then

$$|H(f)|^2 = \frac{-a^2f^2+b^2}{(f-f_1)(f-f_2)(f+f_1)(f+f_2)}.$$

To evaluate the integral we utilize complex contour integration. We consider

$$\oint \frac{-a^2z^2+b^2}{(z-f_1)(z-f_2)(z+f_1)(z+f_2)}dz = 2\pi j \sum[\text{residues inside C}]$$

where the integration is counter clock wise along the curve C where C extends from $-R$ to R along the real axis and is enclosed by a circular are of radius R in the upper z plane. If f_1 and f_2 lie in this region of integration then we evaluate the residues using these two points. We find

$$\begin{aligned}\text{Res}[z=f_1] &= \frac{-a^2f_1^2+b^2}{(f_1-f_2)(f_1+f_1)(f_1+f_2)} \\ &= \frac{-a^2f_1^2+b^2}{\left(\sqrt{c^2-4b}\right)\left(-c+\sqrt{c^2-4b}\right)(-c)}\end{aligned}$$

and

$$\begin{aligned}\text{Res}[z=f_2] &= \frac{-a^2f_2^2+b^2}{(f_2-f_1)(f_2+f_1)(f_2+f_2)} \\ &= \frac{-a^2f_2^2+b^2}{\left(-\sqrt{c^2-4b}\right)(-c)\left(-c-\sqrt{c^2-4b}\right)}\end{aligned}$$

Using the usual arguments we can easily show that the integral over the upper semi-circle tends to zero as $R \to \infty$ since the denominator is of degree 4 and the numerator is of degree 2. So we can write

$$\begin{aligned}B_{eq} &= \frac{1}{G}\int_0^{\infty}|H(f)|^2 df \\ &= \frac{1}{2}\frac{1}{G}\int_{-\infty}^{\infty}|H(f)|^2 df \\ &= \frac{1}{2}\frac{1}{G}2\pi j \sum[\text{residues inside C}] \\ &= \frac{1}{2}\frac{1}{G}2\pi j\left(\text{Res}[z=f_1]+\text{Res}[z=f_2]\right)\end{aligned}$$

Substituting the above residue results and scaling by G we get (after some algebraic manipulation) the desired result. Note that because of symmetry we get the same result if $-f_1$ and $-f_2$ lie in the region of integration.

Problem 3. Suppose the input to the PLL is a signal of the form

$$s(t) = A_c \cos[2\pi f_c t + \phi(t)]$$

and it is corrupted by additive narrowband noise

$$n(t) = x(t) \cos 2\pi f_c t - y(t) \sin 2\pi f_c t$$

where $x(t)$ and $y(t)$ are independent stationary Gaussian noise processes with (two-sided) power spectral density $N_0/2$ W/Hz. This may be written as

$$n(t) = n_i(t) \cos[2\pi f_c t + \phi(t)] - n_q(t) \sin[2\pi f_c t + \phi(t)]$$

where

$$\begin{aligned} n_i(t) &= x(t) \cos \phi(t) + y(t) \sin \phi(t) \\ n_q(t) &= -x(t) \sin \phi(t) + y(t) \cos \phi(t). \end{aligned}$$

After $s(t) + n(t)$ is multiplied by the output of the VCO and the double frequency terms are neglected (since the loop filter filters these out) show that we get the error signal

$$e(t) = A_c \sin \Delta\phi + n_i(t) \sin \Delta\phi - n_q(t) \cos \Delta\phi$$

where $\Delta\phi = \hat{\phi} - \phi$.

Solution: We have

$$\begin{aligned} [s(t) + n(t)][\hat{s}(t)] &= (A_c \cos[2\pi f_c t + \phi(t)] \sin[2\pi f_c t + \hat{\phi}(t)] \\ &\quad + n_i(t) \cos[2\pi f_c t + \phi(t)] \sin[2\pi f_c t + \hat{\phi}(t)] \\ &\quad - n_q(t) \sin[2\pi f_c t + \phi(t)] \sin[2\pi f_c t + \hat{\phi}(t)] \\ &= \frac{1}{2} A_c \left[\sin[4\pi f_c t + \phi(t) + \hat{\phi}(t)] - \sin[\phi(t) - \hat{\phi}(t)]\right] \\ &\quad + \frac{1}{2} n_i(t) \left[\sin[4\pi f_c t + \phi(t) + \hat{\phi}(t)] - \sin[\phi(t) - \hat{\phi}(t)]\right] \\ &\quad - \frac{1}{2} n_q(t) \left[-\cos[4\pi f_c t + \phi(t) + \hat{\phi}(t)] + \cos[\phi(t) - \hat{\phi}(t)]\right]. \end{aligned}$$

Ignoring double frequency terms since they are filtered out we get

$$\begin{aligned} e(t) &= -\frac{1}{2}A_c \sin[\phi(t) - \hat{\phi}(t)] \\ &\quad -\frac{1}{2}n_i(t)\sin[\phi(t) - \hat{\phi}(t)] \\ &\quad -\frac{1}{2}n_q(t)\cos[\phi(t) - \hat{\phi}(t)] \end{aligned}$$

or after absorbing the factor of 1/2 into K in the PLL we get

$$e(t) = A_c \sin \Delta\phi + n_i(t) \sin \Delta\phi - n_q(t) \cos \Delta\phi$$

as desired.