EE 564
Homework 7 Solutions

Problem 1. In class we had

$$\begin{aligned} r_n &= \int_{T_0} r(t)\phi_n(t)dt \\ s_n(\theta) &= \int_{T_0} s(t;\theta)\phi_n(t)dt \end{aligned}$$

where $r(t)$ is the received waveform and $s(t;\theta)$ is the signal waveform. Show

$$\lim_{N\to\infty} \frac{1}{2\sigma^2}\sum_{n=1}^{N}[r_n - s_n(\theta)]^2 = \frac{1}{N_0}\int_{T_0}[r(t)-s(t;\theta)]^2dt.$$

Solution: We note

$$\begin{aligned} r(t) &= \lim_{N\to\infty}\sum_{n=1}^{N} r_n\phi_n(t) \\ s(t;\theta) &= \lim_{N\to\infty}\sum_{n=1}^{N} s_n(\theta)\phi_n(t) \end{aligned}$$

so

$$\begin{aligned} &\frac{1}{N_0}\int_{T_0}[r(t)-s(t;\theta)]^2dt \\ &= \lim_{N\to\infty}\frac{1}{N_0}\int_{T_0}\left[\sum_{n=1}^{N}(r_n - s_n(\theta))\phi_n(t)\right]^2 dt \\ &= \lim_{N\to\infty}\frac{1}{N_0}\int_{T_0}\left[\sum_{n=1}^{N}(r_n - s_n(\theta))\phi_n(t)\right]\left[\sum_{m=1}^{N}(r_m - s_m(\theta))\phi_m(t)\right] dt \\ &= \lim_{N\to\infty}\frac{1}{N_0}\int_{T_0}\sum_{n=1}^{N}\sum_{m=1}^{N}(r_n - s_n(\theta))(r_m - s_m(\theta))\phi_n(t)\phi_m(t)dt \\ &= \lim_{N\to\infty}\frac{1}{N_0}\sum_{n=1}^{N}\sum_{m=1}^{N}(r_n - s_n(\theta))(r_m - s_m(\theta))\delta_{mn} \\ &= \lim_{N\to\infty}\frac{1}{N_0}\sum_{n=1}^{N}(r_n - s_n(\theta))(r_n - s_n(\theta)) \\ &= \lim_{N\to\infty}\frac{1}{2\sigma^2}\sum_{n=1}^{N}[r_n - s_n(\theta)]^2 \end{aligned}$$

where $\sigma^2 = \frac{N_0}{2}$.

Problem 2. In class we derived for an amplitude modulated signal of the form

$$s(t) = A(t)\cos(2\pi f_c t + \phi)$$

and a carrier reference

$$c(t) = \cos(2\pi f_c t + \hat{\phi})$$

the power reduction seen after the demodulator is $\cos^2(\phi - \hat{\phi})$. Plot this power reduction (in dB) as a function of

$$\Delta\phi = |\phi - \hat{\phi}|.$$

Solution: See below for a plot.

Problem 3. Consider a received waveform of the form

$$r(t) = A\cos(2\pi f_c t + \phi) + n(t)$$

where ϕ is an unknown phase. Find both implicit and explicit expressions for $\hat{\phi}_{ML}$ and draw block diagrams illustrating the logic for each expression.

Solution: We seek the value of ϕ, call it $\hat{\phi}_{ML}$, that maximizes

$$\begin{aligned}\Delta_L(\phi) &= \frac{2}{N_0}\int_{T_0} r(t)s(t;\phi)dt \\ &= \frac{2A}{N_0}\int_{T_0} r(t)\cos(2\pi f_c t + \phi)dt.\end{aligned}$$

Setting the derivative equal to 0 yields

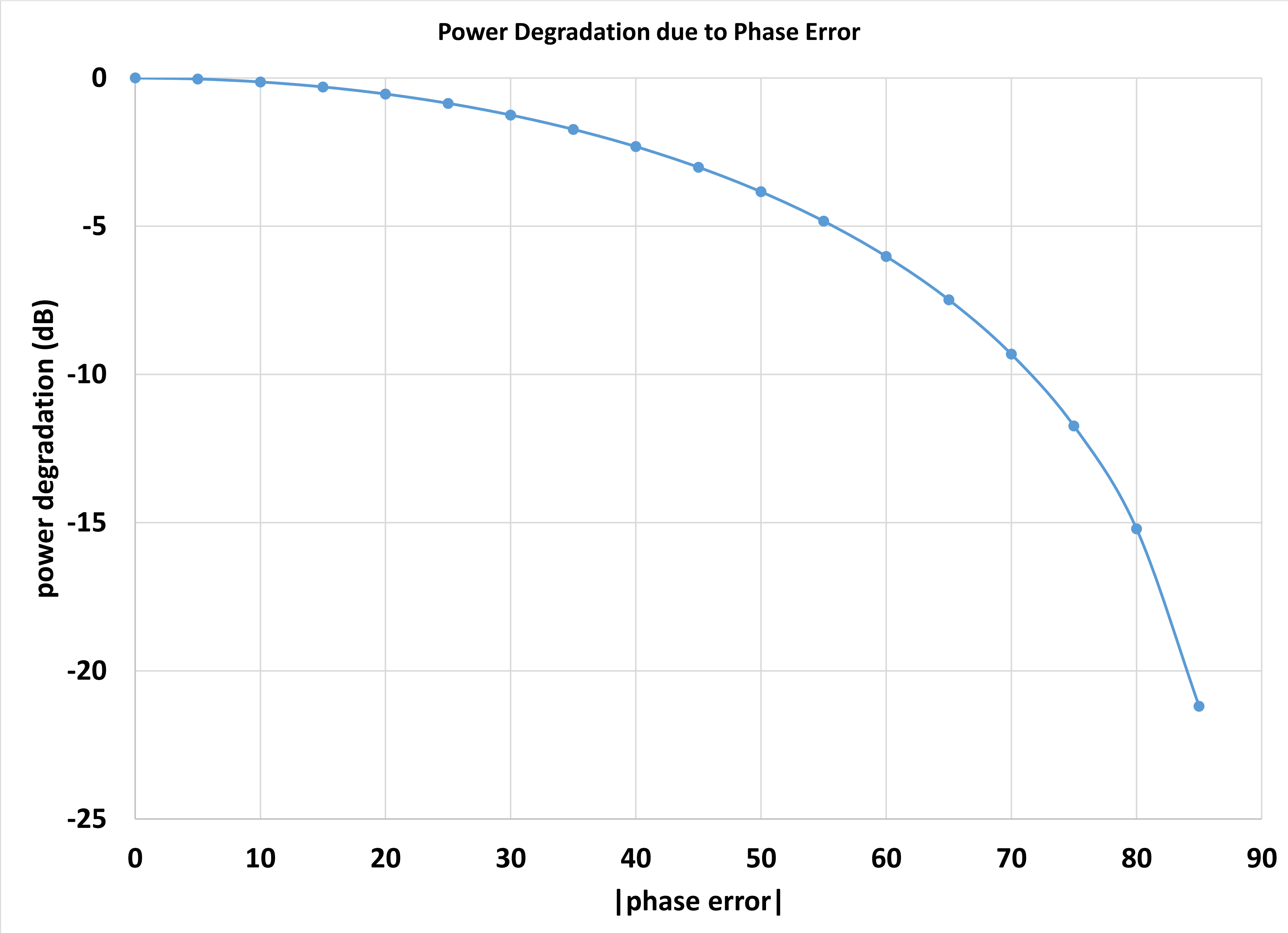
$$\int_{T_0} r(t)\sin(2\pi f_c t + \hat{\phi}_{ML})dt = 0.$$

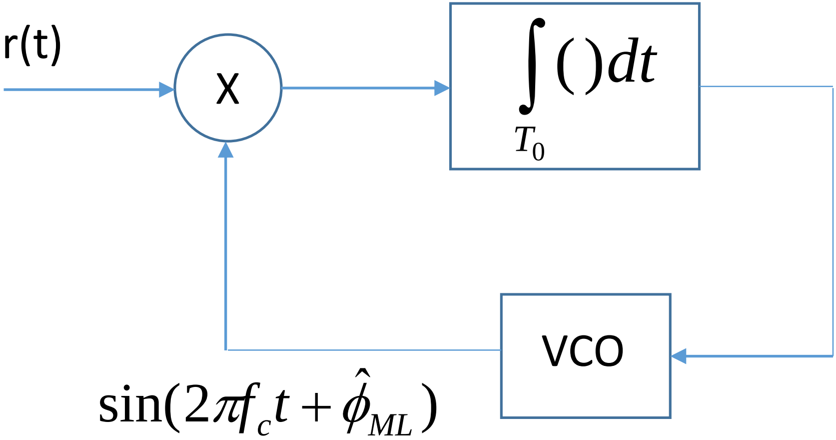
This last result is an implicit expression for $\hat{\phi}_{ML}$. See below for a block diagram.

To obtain an explicit expression we will solve for $\hat{\phi}_{ML}$ from the implicit expression. We get

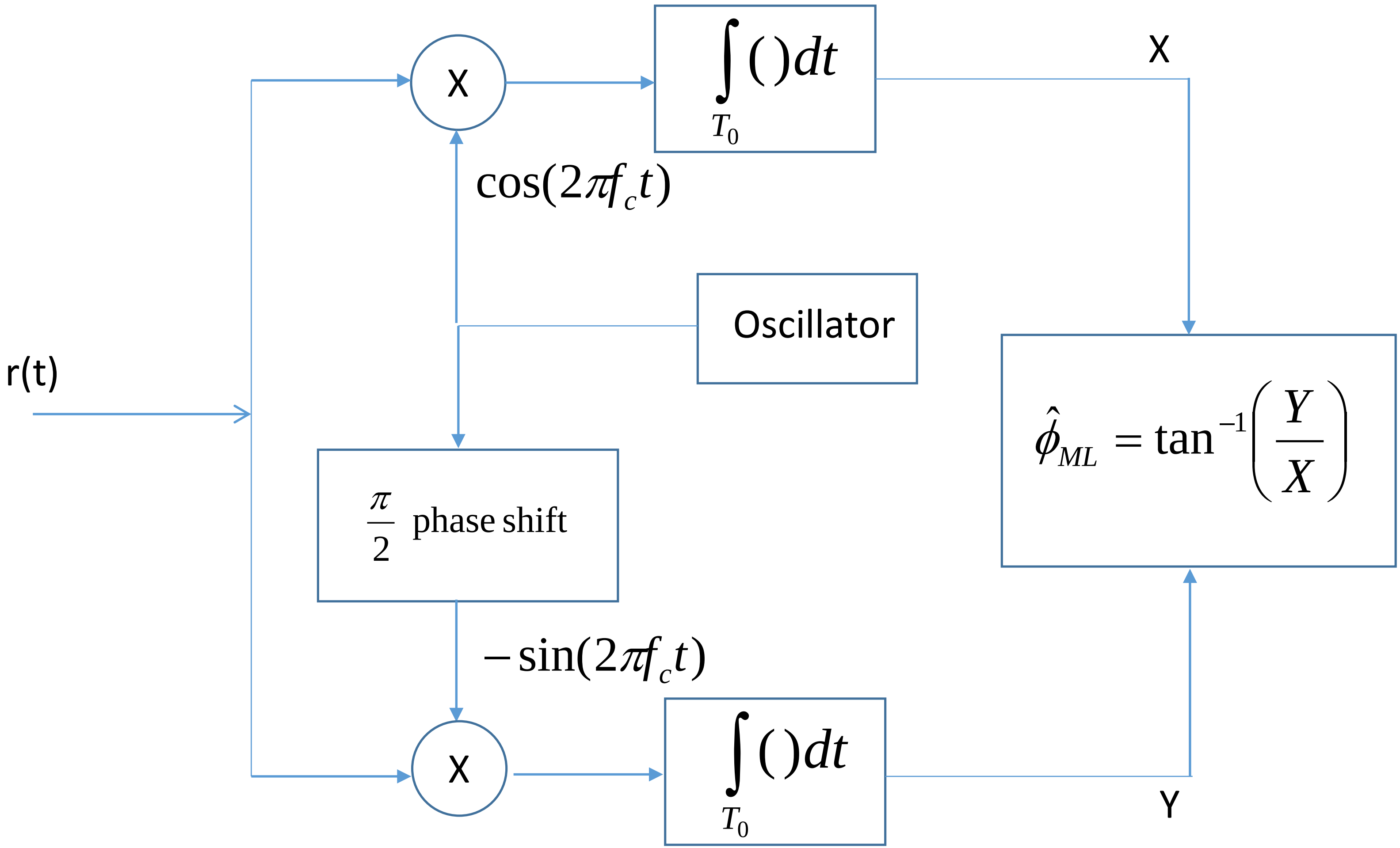
$$\begin{aligned}
&\int_{T_0} r(t)\sin(2\pi f_c t + \hat{\phi}_{ML})dt = 0 \\
&\quad\Rightarrow\quad \int_{T_0} r(t)\left[\sin(2\pi f_c t)\cos\hat{\phi}_{ML} + \cos(2\pi f_c t)\sin\hat{\phi}_{ML}\right]dt = 0 \\
&\quad\Rightarrow\quad \cos\hat{\phi}_{ML}\int_{T_0} r(t)\sin(2\pi f_c t)dt - \sin\hat{\phi}_{ML}\int_{T_0} r(t)\cos(2\pi f_c t)dt = 0 \\
&\quad\Rightarrow\quad \tan\hat{\phi}_{ML} = -\frac{\int_{T_0} r(t)\sin(2\pi f_c t)dt}{\int_{T_0} r(t)\cos(2\pi f_c t)dt} \\
&\quad\Rightarrow\quad \hat{\phi}_{ML} = \tan^{-1}\left(-\frac{\int_{T_0} r(t)\sin(2\pi f_c t)dt}{\int_{T_0} r(t)\cos(2\pi f_c t)dt}\right).
\end{aligned}$$

See below for a block diagram.


Power Degradation due to Phase Error
0
-5
-10
-15
-20
-25
power degradation (dB)
0
10
20
30
40
50
60
70
80
90
|phase error|



HW7 Problem 3 (PLL implicit implementation)



HW7 Problem 3 (explicit implementation)