EE 564

Homework 5 Solutions

Problem 1. In class we looked at binary signaling in an AWGN channel. We let

$$r(t) = \alpha e^{-j\phi} u_1(t) + z(t),$$

that is, we assumed at first that signal 1 was sent. The power spectral density of $z(t)$ is N_0. We defined the complex valued correlation coefficient as

$$\rho = \rho_r + j\rho_i = \frac{1}{2E} \int_0^T u_1(t) u_2^*(t) dt$$

where E denotes energy. When deriving the probability of bit error we had an expression of the form

$$N_m = e^{j\phi} \int_0^T z(t) u_m^*(t) dt$$

and we let $N_{mr} = \Re(N_m)$. The noise components N_{mr}, $m = 1, 2$ are jointly Gaussian with zero mean. We let $V = U_1 - U_2$, where

$$\begin{aligned} U_1 &= 2\alpha E + N_{1r}, \\ U_2 &= 2\alpha E \rho_r + N_{2r}. \end{aligned}$$

a. Show that $E[V] = 2\alpha E(1 - \rho_r)$.

Solution:

$$E[V] = E[U_1 - U_2] = E[U_1] - E[U_2] = 2\alpha E - 2\alpha E \rho_r = 2\alpha E(1 - \rho_r).$$

b. Show that $\sigma_v^2 = 4EN_0(1 - \rho_r)$.

Solution:

$$\begin{aligned} \sigma_v^2 &= E[(N_{1r} - N_{2r})^2] \\ &= E[N_{1r}^2] - 2E[N_{1r}N_{2r}] + E[N_{2r}^2]. \end{aligned}$$

Now

$$\begin{aligned} N_{1r} &= \mathrm{Re}[N_1] = \frac{1}{2}(N_1 + N_1^*) \\ &= \frac{1}{2}\left[e^{j\phi}\int_0^T z(t)u_1^*(t)dt + e^{-j\phi}\int_0^T z^*(t)u_1(t)dt\right]. \end{aligned}$$

Thus,

$$\begin{aligned} 2N_{1r}^2 &= e^{j\phi}\int_0^T z(t)u_1^*(t)dt \cdot e^{j\phi}\int_0^T z(s)u_1^*(s)ds \\ &+ 2e^{j\phi}\int_0^T z(t)u_1^*(t)dt \cdot e^{-j\phi}\int_0^T z^*(s)u_1(s)ds \\ &+ e^{-j\phi}\int_0^T z^*(t)u_1(t)dt \cdot e^{-j\phi}\int_0^T z^*(s)u_1(s)ds \end{aligned}$$

or

$$\begin{aligned} 2N_{1r}^2 &= e^{j2\phi}\int_0^T\int_0^T z(t)z(s)u_1^*(t)u_1^*(s)dtds \\ &+ 2\int_0^T\int_0^T z(t)z^*(s)u_1^*(t)u_1(s)dtds \\ &+ e^{-j2\phi}\int_0^T\int_0^T z^*(t)z^*(s)u_1(t)u_1(s)dtds. \end{aligned}$$

Hence,

$$\begin{aligned} 2E[N_{1r}^2] &= e^{j2\phi}\int_0^T\int_0^T E[z(t)z(s)]u_1^*(t)u_1^*(s)dtds \\ &+ 2\int_0^T\int_0^T E[z(t)z^*(s)]u_1^*(t)u_1(s)dtds \\ &+ e^{-j2\phi}\int_0^T\int_0^T E[z^*(t)z^*(s)]u_1(t)u_1(s)dtds. \end{aligned}$$

Now $E[z(t)z(s)] = 0$ if $t \neq s$ since we have white noise. For $t = s$ write

$$z(t) = x(t) + jy(t)$$

so that

$$\begin{aligned} E[z(t)z(t)] &= E[x^2(t)] - E[y^2(t)] + 2E[x(t)y(t)] \\ &= \phi_{xx}(0) - \phi_{yy}(0) + 2\phi_{xy}(0). \end{aligned}$$

From a previous homework we know $\phi_{xx}(\tau) = \phi_{yy}(\tau)$ (so $\phi_{xx}(0) - \phi_{yy}(0) = 0$) and $\phi_{xy}(\tau) = -\phi_{yx}(\tau)$ (so $\phi_{xy}(0) = 0$). Hence, $E[z(t)z(t)] = 0$. We have thus shown

$$E[z(t)z(s)] = 0$$

for all t, s. A similar argument also implies

$$E[z^*(t)z^*(s)] = 0.$$

Therefore,

$$2E[N_{1r}^2] = 2\int_0^T \int_0^T N_0\delta(t-s)u_1^*(t)u_1(s)dtds$$

which becomes

$$\begin{aligned} 2E[N_{1r}^2] &= 2N_0\int_0^T u_1^*(t)u_1(t)dt \\ &= 4N_0E. \end{aligned}$$

Hence,

$$E[N_{1r}^2] = 2N_0E.$$

Similarly,

$$E[N_{2r}^2] = 2N_0E.$$

We also find in a similar manner that

$$2E[N_{1r}N_{2r}] = 2N_0\int_0^T u_1^*(t)u_2(t)dt.$$

Hence,

$$2E[N_{1r}N_{2r}] = 4N_0E\rho_r.$$

Putting all of this together then gives

$$\sigma_v^2 = 4EN_0(1-\rho_r).$$

Problem 2. For the binary signal we derived

$$P_b = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{\alpha^2 E}{2N_0}(1-\rho_r)}\right)$$

where

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt.$$

Write P_b in term of the Gaussian tail function

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-u^2/2}du.$$

Solution: Let $u = \sqrt{2}t$. Then,

$$\begin{aligned}\text{erfc}(x) &= \frac{2}{\sqrt{\pi}}\int_{\sqrt{2}x}^\infty e^{-u^2/2}\frac{1}{\sqrt{2}}du \\ &= \sqrt{\frac{2}{\pi}}\int_{\sqrt{2}x}^\infty e^{-u^2/2}du \\ &= \frac{2}{\sqrt{2\pi}}\int_{\sqrt{2}x}^\infty e^{-u^2/2}du \\ &= 2Q(\sqrt{2}x).\end{aligned}$$

Problem 3. Suppose we wish to compute the probabilty of bit error using a simulation. We run the simulation for n trials and count X errors. Our estimate for the probability of bit error, P_b, is

$$\hat{P}_b = \frac{X}{n}.$$

We may regard X as a sum of Bernoulli trials (so X is a binomail random variable) with the result that

$$E[X] = nP_b, \quad Var[X] = nP_b(1-P_b).$$

a. Show $E[\hat{P}_b] = P_b$.

Solution:

$$E[\hat{P}_b] = \frac{E[X]}{n} = \frac{nP_b}{n} = P_b.$$

b. Show $Var[\hat{P}_b] = \frac{P_b(1-P_b)}{n}$.

Solution:

$$Var[\hat{P}_b]] = \frac{Var[X]}{n^2} = \frac{nP_b(1-P_b)}{n^2} = \frac{P_b(1-P_b)}{n}.$$

We can make use of the Central Limit Theorem and use the approximation

$$\frac{\hat{P}_b - E[\hat{P}_b]}{\sqrt{Var[\hat{P}_b]}} \sim N(0,1)$$

provided $X \geq 5$ and $n - X \geq 5$. Thus, under these conditions

$$\frac{\hat{P}_b - P_b}{\sqrt{Var[\hat{P}_b]}} \sim N(0,1).$$

Now 95% of the time we will have

$$-1.96 < \frac{\hat{P}_b - P_b}{\sqrt{Var[\hat{P}_b]}} < 1.96$$

and thus 95% of the time

$$\hat{P}_b - \tilde{E} < P_b < \hat{P}_b + \tilde{E}$$

where,

$$\begin{aligned} \tilde{E} &= 1.96\sqrt{Var[\hat{P}_b]} \\ &= 1.96\sqrt{\frac{P_b(1-P_b)}{n}}. \end{aligned}$$

However, since P_b is unknown we take the margin of error as

$$E = 1.96\sqrt{\frac{\hat{P}_b(1-\hat{P}_b)}{n}}.$$

Hence, a 95% confidence interval (CI) for P_b is

$$95\%\ \text{CI} = \hat{P}_b \pm E.$$

c. Suppose in your simulation you count 100 errors out of 100,000 trials. Compute your estimate for the probability of bit error and construct a 95% CI about it using the above formulas.

Solution: We find

$$\hat{P}_b = \frac{100}{100000} = 10^{-3}.$$

The margin of error is

$$E = 1.96 \times 10^{-4}.$$

Hence, the 95% CI $= \hat{P}_b \pm E$ is the interval

$$(8.04 \times 10^{-4},\ 1.20 \times 10^{-3}).$$

d. Now suppose you decide to utilize the actual underlying binomial distribution to construct a CI. Again, assuming you count 100 errors out of 100,000 trials construct a 95% CI for the probability of bit error (just like before you can substitute $\hat{P}_b$ for P_b in your calculations).

Solution: We compute by trial substitutions that 95% of time the number of erros counted will be between 80 and 100, that is,

$$P(80 \le X \le 120) = \sum_{k=80}^{120} \binom{N}{k} \hat{P}_b^k (1-\hat{P}_b)^{N-k} = 0.97$$

where $N = 100000$, and this is as close as we can get to being at least 95% with symmetry about $X = 100$. Hence, the 95% CI $= \hat{P}_b \pm E$ is the interval

$$(8.00 \times 10^{-4},\ 1.20 \times 10^{-3}).$$

So the two approaches we took agree quite closely.

e. Sometimes we want to be confident that the actual probability of bit error is not greater than what we estimate (this would arise, for instance, when verifying some performance specification for a system). Using the binomial distribution what is the maximum value of P_b such that 95% of the time the observed number of bit errors is 100 or less out of 100,000 trials? (Since we observe 100 errors then we can be 95% confident that the true P_b is not greater than this P_b we just computed since if it were greater then we would have observed something that would happen less than 5% of the time).

Solution: We find with $\tilde{P}_b = 8.50 \times 10^{-4}$ that

$$P(X \leq 100) = \sum_{k=0}^{100} \binom{N}{k} \tilde{P}_b^k (1 - \tilde{P}_b)^{N-k} = 0.95.$$