EE 564

Homework 4 Solutions

Problem 1. In class we showed

$$\begin{aligned} E[n(t)n(t+\tau)] &= \phi_{xx}(\tau)\cos 2\pi f_c t \cos 2\pi f_c(t+\tau) \\ &\quad +\phi_{yy}(\tau)\sin 2\pi f_c t \sin 2\pi f_c(t+\tau) \\ &\quad -\phi_{xy}(\tau)\sin 2\pi f_c t \cos 2\pi f_c(t+\tau) \\ &\quad -\phi_{yx}(\tau)\cos 2\pi f_c t \sin 2\pi f_c(t+\tau) \end{aligned}$$

and concluded

$$\phi_{nn}(\tau) = \phi_{xx}(\tau)\cos 2\pi f_c \tau - \phi_{yx}(\tau)\sin 2\pi f_c \tau.$$

Show how the last result is obtained from the prior expression.

Solution: Using

$$\begin{aligned} \cos A \cos B &= \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right] \\ \sin A \sin B &= \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] \\ \sin A \cos B &= \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right] \end{aligned}$$

the above becomes

$$\begin{aligned} E[n(t)n(t+\tau)] &= \frac{1}{2}\left[\phi_{xx}(\tau) + \phi_{yy}(\tau)\right]\cos 2\pi f_c \tau \\ &+ \frac{1}{2}\left[\phi_{xx}(\tau) - \phi_{yy}(\tau)\right]\cos 2\pi f_c(2t+\tau) \\ &- \frac{1}{2}\left[\phi_{yx}(\tau) - \phi_{xy}(\tau)\right]\sin 2\pi f_c \tau \\ &- \frac{1}{2}\left[\phi_{yx}(\tau) + \phi_{xy}(\tau)\right]\sin 2\pi f_c(2t+\tau). \end{aligned}$$

Since $n(t)$ is stationary the right-hand side of the last expression must be independent ot t. Thus, we must have $\phi_{xx}(\tau) = \phi_{yy}(\tau)$ and $\phi_{xy}(\tau) = -\phi_{yx}(\tau)$. Hence, we get

$$\phi_{nn}(\tau) = \phi_{xx}(\tau)\cos 2\pi f_c \tau - \phi_{yx}(\tau)\sin 2\pi f_c \tau$$

which was to be shown.

Problem 2. Show that

$$\Phi_{nn}(f) = \int_{-\infty}^{\infty} \left[\mathrm{Re}(\phi_{zz}(\tau) e^{j2\pi f_c \tau}) \right] e^{-j2\pi f \tau} d\tau$$

becomes

$$\Phi_{nn}(f) = \frac{1}{2} \left[\Phi_{zz}(f - f_c) + \Phi_{zz}(-f - f_c) \right].$$

Solution: Using

$$\mathrm{Re}(z) = \frac{1}{2}(z + z^*)$$

we get

$$\begin{aligned}
\Phi_{nn}(f) &= \frac{1}{2} \int_{-\infty}^{\infty} \left[\phi_{zz}(\tau) e^{j2\pi f_c \tau} + \phi_{zz}^*(\tau) e^{-j2\pi f_c \tau} \right] e^{-j2\pi f \tau} d\tau \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \phi_{zz}(\tau) e^{j2\pi f_c \tau} e^{-j2\pi f \tau} d\tau + \int_{-\infty}^{\infty} + \phi_{zz}^*(\tau) e^{-j2\pi f_c \tau} e^{-j2\pi f \tau} d\tau \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \phi_{zz}(\tau) \tau e^{-j2\pi (f - f_c) \tau} d\tau + \int_{-\infty}^{\infty} \phi_{zz}^*(\tau) \tau e^{-j2\pi (f + f_c) \tau} d\tau \\
&= \frac{1}{2} \left[\Phi_{zz}(f - f_c) + \Phi_{zz}^*(-f - f_c) \right] \\
&= \frac{1}{2} \left[\Phi_{zz}(f - f_c) + \Phi_{zz}(-f - f_c) \right]
\end{aligned}$$

since Φ_{zz} is real.

Problem 3. We stated in class that the equivalent lowpass noise $z(t)$ has a power spectral density

$$\Phi_{zz}(f) = \begin{cases} N_0, & |f| \leq B/2 \\ 0, & |f| > B/2. \end{cases}$$

Show that the corresponding autocorrelation function is

$$\phi_{zz}(\tau) = N_0 \frac{\sin \pi B \tau}{\pi \tau}.$$

Solution:

$$\begin{aligned}\phi_{zz}(\tau) &= \int_{-\infty}^{\infty} \Phi_{zz}(f) e^{j2\pi f\tau} df \\ &= \int_{-B/2}^{B/2} N_0 e^{j2\pi f\tau} df \\ &= N_0 \frac{1}{j2\pi\tau}\left(e^{j2\pi B/2\tau} - e^{-j2\pi B/2\tau}\right) \\ &= N_0 \frac{\sin \pi B\tau}{\pi\tau}.\end{aligned}$$

Problem 4. Hilbert transforms are sometimes encountered in the analysis of narrowband systems. Given a real-valued function $x(t)$, its Hilbert transform, denoted $\hat{x}(t)$, is defined by

$$\hat{x}(t) = x(t) * h(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau$$

where

$$h(t) = \frac{1}{\pi t}$$

with corresponding frequency response

$$H(f) = \begin{cases} -j, & f > 0 \\ 0, & f = 0 \\ j, & f < 0. \end{cases}$$

The integral above is to be interpreted as the principal value of the integral, that is, for a function $g(t)$

$$\int_{-\infty}^{\infty} g(t) dt$$

means

$$\lim_{\epsilon \to 0}\left[\int_{-\infty}^{-\epsilon} g(t)dt + \int_{\epsilon}^{\infty} g(t)dt\right].$$

a. Show $\hat{\hat{x}}(t) = -x(t)$, that is, the Hilbert transform of the Hilbert transform of $x(t)$ equals $-x(t)$.

Solution: We may write

$$H(f) = -j \cdot sgn(f)$$

and note that $\hat{\hat{x}}(t)$ results from passing $x(t)$ thru a filter with this response. Hence, a double Hilbert transform results from passing $x(t)$ thru a cascade of two such filters with transfer function

$$H^2(f) = [-j \cdot sgn(f)]^2 = -1$$

yielding $\hat{\hat{x}}(t) = -x(t)$.

b. Show the Hilbert transform of $x(t) = \cos(\omega t + \phi)$ is $\hat{x}(t) = \sin(\omega t + \phi)$.

Solution:

$$\begin{aligned}\hat{x}(t) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos(\omega\tau + \phi)}{t-\tau} d\tau \\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos(\omega(u+t)+\phi)}{u} du\end{aligned}$$

which becomes

$$\hat{x}(t) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos(\omega t + \phi)\cos\omega u}{u} du + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\omega t+\phi)\sin\omega u}{u} du.$$

The principal value of the first integral is zero since the integrand is odd. Therefore,

$$\hat{x}(t) = \frac{\sin(\omega t + \phi)}{\pi}\int_{-\infty}^{\infty} \frac{\sin\omega u}{u} du.$$

But,

$$\int_{-\infty}^{\infty} \frac{\sin\omega u}{u} du = \pi$$

so

$$\hat{x}(t) = \sin(\omega t + \phi).$$

c. Using [a] and [b] show that the Hilbert transform of $y(t) = \sin(\omega t + \phi)$ is $\hat{y}(t) = -\cos(\omega t + \phi)$.

Solution: Let $x(t) = \cos(\omega t + \phi)$. Then,

$$\hat{x}(t) = \sin(\omega t + \phi) = y(t).$$

Hence,

$$\hat{y}(t) = \hat{\hat{x}}(t) = -x(t) = -\cos(\omega t + \phi).$$