EE 564

Homework 1 Solutions

Problem 1. The continuous random variable X has *pdf*

$$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

a. Find $P(X \leq 2)$.

Solution:

$$\begin{aligned} P(X \leq 2) &= \int_0^2 2e^{-2x} \\ &= 1 - e^{-4} \\ &= 0.98. \end{aligned}$$

b. Find $P(X \leq 2 | X > 1)$.

Solution:

$$\begin{aligned} P(X \leq 2 | X > 1)) &= \frac{P(1 < X \leq 2)}{P(X > 1)} \\ &= \frac{\int_1^2 2e^{-2x}}{\int_1^\infty 2e^{-2x}} \\ &= \frac{e^{-2} - e^{-4}}{e^{-2}} \\ &= 0.86. \end{aligned}$$

Problem 2. Suppose the normal random variable X has mean 1 and variance 4. Let $Y = X^2 + 1$. Find the mean and variance of Y.

Solution:

$$E[Y] = E[X^2 + 1] = E[X^2] + 1 = \sigma_X^2 + \mu_X^2 = (4 + 1) + 1 = 6.$$

Problem 3. Suppose $X \sim N(2,\ 3)$,i.e., X is normally distributed with mean 2 and variance 3. Find the value of b that minimizes $E\left[(X - b)^2\right]$.

Solution: The value of b that minimizes $E\left[(X-b)^2\right]$ is

$$b_{opt} = E[X] = 2.$$

Problem 4. Suppose $X \sim N(2,\ 3)$. Find $E\left[X^4\right]$.

Solution: Let $Z = \frac{X-2}{\sqrt{3}}$. Then Z is standard normal. $E\left[Z^4\right] = 3$ and $E[Z^2] = 1$ so

$$\begin{aligned} E\left[X^4\right] &= E\left[(\sqrt{3}Z+2)^4\right] \\ &= E\left[(3Z^2+4\sqrt{3}Z+4)^2\right] \\ &= 9E[Z^4]+48E[Z^2]+16+24E[Z^2] = 115. \end{aligned}$$

Problem 5. Suppose the random variable X has mean 0 and variance 4. Let $Y = 2X+1$. Find the correlation coefficient, r_{XY}.

Solution: Since Y is of the form $Y = aX+b$ with $a > 0$ then $r_{XY} = 1$.

Problem 6. Suppose $X \sim N(1,\ 2)$, and $Y \sim N(0,\ 6)$, and X is independent of Y. Let $Z = X+Y$. Find the variance of Z.

Solution: $Var(Z) = Var(X) + Var(Y) = 8$.

Problem 7. The two-dimensional continuous random variable (X, Y) has joint *pdf*

$$f_{XY}(x,y) = \begin{cases} x+y, & 0 \le x \le 1,\ 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a. Compute $f_{Y|X}(y|x)$.

Solution.

$$f_{Y|X}(y|x) = f(y|x) = \frac{f(x,y)}{f(x)} = \frac{x+y}{f(x)}.$$

Now

$$f(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 (x+y)dy = x + \frac{1}{2}.$$

So

$$f(y|x) = \begin{cases} \dfrac{x+y}{x+1/2}, & 0 \le x \le 1,\ 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

b. Find $E[Y|X = x]$.

Solution.

$$E[Y|X = x] = \int_{-\infty}^{\infty} y f(y|x) dy = \int_0^1 y \frac{x+y}{x+1/2} dy = \frac{3x+2}{6x+3}.$$

Problem 8. Suppose we toss a fair coin 50 times. Let X denote the number of heads obtained.

a. Find $P(22 < X \le 28)$ using the binomial distribution.

Solution.

$$P(22 < X \le 28) = \sum_{k=23}^{28} \binom{50}{k} (1/2)^{50} = 0.5989.$$

b. Find $P(22 < X \le 28)$ using the normal approximation. Write your answer using

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du.$$

Solution.

$$\mu = np = 50(1/2) = 25,$$

$$\sigma^2 = npq = 50(1/2)(1/2) = 12.5 \Rightarrow \sigma = 3.5355.$$

$$P_{binomial}(22 < X \le 28) = P_{normal}\left(\frac{22.5-\mu}{\sigma} < \frac{X-\mu}{\sigma} \le \frac{28.5-\mu}{\sigma}\right)$$

$$\approx P_{normal}\left(-0.7071 < Z \le 0.9899\right)$$

where $Z \sim N(0,\ 1)$. So

$$P_{binomial}(22 < X \le 28) \approx \Phi(0.9899) - \Phi(-0.7071).$$

Note: $\Phi(0.8485) - \Phi(-0.8485) = 0.5992$. We applied continuity correction in our calculations above.