Name: ______________________ **On-campus or DEN student: _________**

EE 567 Midterm Solution

October 29, 2019

Inst: Dr. C.W. Walker

Problem	Points	Score
1	10	
2	12	
3	13	
4	15	
5	12	
6	12	
7	12	
8	14	
Total	100	

Instructions and Information:

1) Print your name and location at the top of the page.

2) Make sure your exam has 8 problems.

3) This is a closed book exam. You may use one 8.5 x 11 inch sheet of handwritten notes (front and back). You may use a self-contained calculator but not a computer. Cell phones or any device with internet capability is not permitted. **You have 1 hour and 20 minutes to take this exam**.

4) The points for each problem is shown above.

Problem 1. A carrier wave of frequency 200 MHz is frequency-modulated by a sine-wave of amplitude 6 volts and frequency 20 kHz. The frequency sensitivity of the modulator is 10 kHz per volt. Determine the approximate bandwidth of the FM wave using Carson's rule.

Solution:

$$B_T = 2\Delta f + 2f_m.$$

$$\Delta f = k_f A_m = 10,000 \times 6 = 60,000.$$

$$B_T = 2 \times 60,000 + 2 \times 20,000 = 160 \text{ kHz}.$$

Problem 2. A certain AM signal is given as

$$s_{AM}(t) = [A + 2\cos(2\pi f_m t)]\cos(2\pi f_c t)$$

where $A > 0$. The value of f_c is much greater than the bandwidth of the signal.

a. What is the modulating signal, $m(t)$?

   Solution: $m(t) = 2\cos(2\pi f_m t)$.

b. What is the modulation index?

   Solution: $\mu = \dfrac{m_p}{A} = \dfrac{2}{A}$, where $m_p = \max m(t) = 2$.

c. What is the minimum value of A that allows envelope detection to be used in recovering $m(t)$ without distortion?

   Solution: $0 \leq \mu \leq 1 \Rightarrow A \geq 2$ so the min $A = 2$.

d. Suppose now that we replace A in the expression for $s_{AM}(t)$ with $\tilde{A}$ where we model $\tilde{A}$ as a normal random variable with mean $\mu = A$ and standard deviation $\sigma = 0.1$. Now what is the minimum value of A that allows envelope detection to be used in recovering $m(t)$ such that we can avoid distortion 97.5% of the time during those time instances of the waveform when distortion is most likely to occur?

Note: You may make use of the fact that if Z is a standard normal random variable then $P(-1.96 < Z < 1.96) = 0.95$.

Solution: Using part (c) we need

$$P(\tilde{A} \geq 2) = 0.975$$

or

$$P\left(\frac{\tilde{A} - A}{\sigma} \geq \frac{2 - A}{\sigma}\right) = 0.975$$

or

$$P\left(Z \geq \frac{2 - A}{\sigma}\right) = 0.975$$

where $Z \sim N(0, 1)$. Using the hint we see that

$$\frac{2 - A}{\sigma} = -1.96$$

from which we get $A = 2.196$.

Problem 3. In class we considered an ideal (no noise) reception of a BPSK signal

$$r(t) = A\cos(2\pi f_c t + \phi)$$

where A has the same sign for T seconds and we just set $\phi = 0$ in our classroom example. We showed how to process this signal to determine the information (i.e., the sign of A) by mixing this signal with a locally generated signal and then integrating for T seconds before making our decision. Suppose at the receiver ϕ may not be 0 and we do not know the exact value of ϕ so we use $\phi + \phi_e$ for the phase in the local mixing signal. Hence, ϕ_e represents a phase error. If $T = 2$ second find the largest value of ϕ_e that would still permit us to make a correct decision regarding the sign of A.

Note: You may assume that $f_c \gg 1/T$ so any terms containing sinusoids with frequencies that are multiplies of f_c that are integrated may be ignored (they are filtered out as shown in class).

Solution: We compute

$$A\cos(2\pi f_c t + \phi) \cdot \cos(2\pi f_c t + \phi + \phi_e)$$
$$= \frac{A}{2}\cos(\phi_e) + \text{ higher frequency terms}$$

Integrating for T seconds (which acts as a LPF) we get

$$\frac{AT}{2}\cos(\phi_e)$$

which when evaluated at $T = 2$ becomes

$$A\cos(\phi_e).$$

To keep this the proper sign we require $\cos(\phi_e) > 0$ so $\phi_e < \frac{\pi}{2}$.

Problem 4. This problem deals with single-sideband modulation. For a signal $s(t)$ let $\hat{s}(t)$ denote the Hilbert transform of $s(t)$ and let

$$\begin{aligned} s_+(t) &= \frac{1}{2}\left[s(t) + j\hat{s}(t)\right] \\ s_-(t) &= \frac{1}{2}\left[s(t) - j\hat{s}(t)\right] \end{aligned}$$

be two complex-values signals associated with $s(t)$.

a. Show that the Fourier transform of $s_+(t)$ is

$$S_+(f) = S(f)u(f)$$

where $u(f)$ is the unit step function taking the value of unity for $f \geq 0$ and 0 elsewhere.

Note: Similarly, one can show that the Fourier transform of $s_-(t)$ is

$$S_-(f) = S(f)u(-f)$$

but you do not need to show this result.

Solution:

$$\begin{aligned} S_+(f) &= \frac{1}{2}\left[S(f) + j\hat{S}(f)\right] \\ &= \frac{1}{2}\left[S(f) - j^2\text{sgn}(f)S(f)\right] \\ &= S(f)\frac{1}{2}\left[1 + \text{sgn}(f)\right] \\ &= S(f)u(f). \end{aligned}$$

b. Now let $s(t)$ be a lowpass modulating signal bandlimited to W Hz and let $f_c > W$ be the carrier frequency. Let $s_{USB}(t)$ be the upper sideband signal after frequency translation. Give an expression for $S_{USB}(f)$ using frequency translations of $S_+(f)$ and $S_-(f)$.

Solution:

$$S_{USB}(f) = S_+(f - f_c) + S_-(f + f_c).$$

c. Same conditions as part (b). Let $s_{LSB}(t)$ be the lower sideband signal. Give an expression for $S_{LSB}(f)$ using frequency translations of $S_+(f)$ and $S_-(f)$.

Solution:

$$S_{LSB}(f) = S_-(f - f_c) + S_+(f + f_c).$$

d. Derive an expression for $s_{USB}(t)$ in terms of $s(t)$ and $\hat{s}(t)$.

Solution:

$$\begin{aligned}
s_{USB}(t) &= s_+(t)e^{j2\pi f_c t} + s_-(t)e^{-j2\pi f_c t} \\
&= \frac{1}{2}\left[s(t) + j\hat{s}(t)\right]e^{j2\pi f_c t} + \frac{1}{2}\left[s(t) - j\hat{s}(t)\right]e^{-j2\pi f_c t} \\
&= s(t)\frac{1}{2}\left[e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right] \\
&\quad +\hat{s}(t)\frac{1}{2}\left[je^{j2\pi f_c t} - je^{-j2\pi f_c t}\right] \\
&= s(t)\cos(2\pi f_c t) - \hat{s}(t)\sin(2\pi f_c t).
\end{aligned}$$

e. Derive an expression for $s_{LSB}(t)$ in terms of $s(t)$ and $\hat{s}(t)$.

Solution: A similar calcuation as above yields

$$s_{LSB}(t) = s(t)\cos(2\pi f_c t) + \hat{s}(t)\sin(2\pi f_c t).$$

Problem 5. Consider a wide-sense stationary random process $X(t)$ with autocorrelation function

$$R_X(\tau) = \begin{cases} 1 - \dfrac{|\tau|}{T}, & |\tau| \le T \\ 0, & |\tau| > T. \end{cases}$$

Find $S_X(f)$, the power spectral density of $X(t)$.

Solution: We recognize that $R_X(\tau)$ is the convolution of a rectangular function with itself. Let

$$Rect(\tau) = \begin{cases} 1, & |\tau| \le T/2 \\ 0, & |\tau| > T/2. \end{cases}$$

Then

$$\begin{aligned} S_X(f) = F[R_X(\tau)] &= F\left[\frac{1}{\sqrt{T}} Rect(\tau) * \frac{1}{\sqrt{T}} Rect(\tau)\right] \\ &= F\left[\frac{1}{\sqrt{T}} Rect(\tau)\right] \cdot F\left[\frac{1}{\sqrt{T}} Rect(\tau)\right]. \end{aligned}$$

Now

$$F\left[\frac{1}{\sqrt{T}} Rect(\tau)\right] = \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} e^{-i2\pi f t} dt = \frac{1}{\sqrt{T}} \frac{\sin \pi f T}{\pi f}.$$

Hence,

$$\begin{aligned} S_X(f) &= \frac{1}{T} \left(\frac{\sin \pi f T}{\pi f}\right)^2 \\ &= T \operatorname{sinc}^2(fT). \end{aligned}$$

Problem 6. An angle modulated signal is described by

$$s(t) = 5\cos\left(2\pi f_c t + 0.2\sin(2\pi f_m t)\right)$$

where $f_c = 1$ MHz and $f_m = 1$ kHz.

a. Find the power of the modulated signal $s(t)$.
Solution:

$$P = \frac{5^2}{2} = 12.5 \text{ W}.$$

b. Find the frequency deviation, Δf.
Solution:

$$\theta(t) = 2\pi f_c t + 0.2\sin(2\pi f_m t)$$

$$f_i(t) = \frac{1}{2\pi}\frac{d\theta(t)}{dt} = f_c + 0.2 f_m \cos(2\pi f_m t)$$

$$\Delta f = 200 \text{ Hz}.$$

Problem 7. In class we considered the linearized model of a PLL via

$$\frac{d\phi_e(t)}{dt} + 2\pi k_0 \int_{-\infty}^{\infty} \phi_e(\tau) h(t-\tau) d\tau = \frac{d\phi_1(t)}{dt}$$

where $\phi_1(t)$ was the input phase to be tracked and $\phi_e(t) = \phi_1(t) - \hat{\phi}_1(t)$ was the resulting phase estimate error. Suppose the transfer function of the loop filter, $H(f)$, is of the form

$$H(f) = \frac{1}{j2\pi f + \sqrt{2}}.$$

Find the closed loop transfer function, that is, find the transfer function relating the Fourier transform of the estimate of $\phi_1(t)$ to the Fourier transform of $\phi_1(t)$.

Solution:

We find

$$j2\pi f \Phi_e(f) + 2\pi k_0 \Phi_e(f) H(f) = j2\pi f \Phi_1(f)$$

and

$$\Phi_e(f) = \frac{1}{1 + L(f)} \Phi_1(f)$$

where

$$L(f) = \frac{k_0 H(f)}{jf}.$$

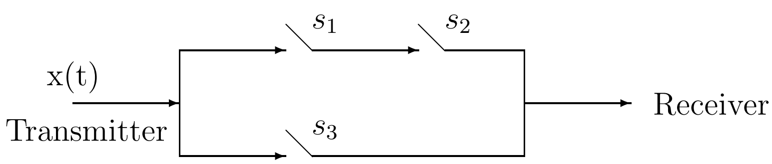
Now

$$\Phi_e(f) = \Phi_1(f) - \hat{\Phi}_1(f)$$

so

$$\begin{aligned} G(f) : &= \frac{\hat{\Phi}_1(f)}{\Phi_1(f)} \\ &= 1 - \frac{1}{1 + L(f)} \\ &= \frac{k_0}{-2\pi f^2 + j\sqrt{2} f + k_0}. \end{aligned}$$

Problem 8. Consider the transmission of a signal as shown in the following diagram.



A signal is transmitted along two paths as shown. In the upper path there are two switches to pass through while in the lower path there is one switch to pass through. Each switch s_i operates independently and allows the signal arriving at the switch to pass with probability p_i for $i = 1, 2, 3$. The signal transmission is successful if the signal $x(t)$ sent at the transmitter reaches the receiver along either or both paths. Find the probability that the transmission is successful if

a. $p_1 = 1/3,\ p_2 = 2/3,\ p_3 = 1/2.$

Solution: Let S denote success. We find

$$P(S) = p_1 p_2 + p_3 - p_1 p_2 p_3 = \frac{11}{18}.$$

b. $p_1 = 1/3$, $p_2 = 2/3$, p_3 is a discrete random variable with probability mass function

$$P\left(p_3 = \frac{3}{4}\right) = \frac{1}{2}, \quad P\left(p_3 = 0\right) = \frac{1}{2}.$$

Solution: We find

$$P(S) = \frac{1}{3} \cdot \frac{2}{3} + \frac{3}{4} \cdot \frac{1}{2} - \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} = \frac{37}{72}.$$