EE 567

Homework 10 Solution

Problem 1. A binary communication system is used where at the receiver the test statistic is $z(T) = ca_i + n_0,\ i = 1, 2$ where $a_1 = +1$ and $a_2 = -1$ and $c = 1$ if a signal is present and $c = 0$ if a signal is not present. The noise n_0 is distributed as

$$p(n_0) = \begin{cases} n_0 + 1.0, \ -1.0 \le n_0 \le 0 \\ \\ -(n_0 - 1.0), \ 0 \le n_0 \le 1.0 \\ \\ 0 \text{ elsewhere.} \end{cases}$$

However, the receiver is just interested in detecting whether or not the signal is actually present. If the signal is not present the noise n_0 is still present and if the signal is present the noise is present also. So, the receiver designer decides to take the absolute value of the test statistic and compare this to a threshold. If $|Z(T)| > T_0$, for some T_0, then a signal is declared present, otherwise, it is not.

a. Find T_0 such that the probability of false alarm is 10^{-4}.

   Solution: A graph of the density function is a triangle centered at zero. An equation of the right side of the absolute value of the density is $y = -2z + 2$. Thus, we solve

   $$\frac{1}{2}(1 - T_0)(-2T_0 + 2) = 10^{-4}$$

   to get $T_0 = 0.99$.

b. If a signal is present find the probability that you would detect it with this logic.

   Solution: With a signal present the pdf $f(z)$ is a triangle centered at ± 1 and after taking absolute values it is a triangle centered at 1 with region of support extending from 0 to 2. The equation of the left side of the triangle is $y = z$. Thus,

   $$P_d = \int_{T_0}^{2} f(z)dz = \int_{0.99}^{1} zdz + \frac{1}{2} = 0.51$$

Problem 2. Suppose we receive a BPSK signal of the form

$$r(t) = A_1 \cos(2\pi f_c t) + n(t), \quad 0 \le t \le T$$

where $n(t)$ is a noise process, $T = 1$ sec, $f_c = 1$ kHz and $A_1 = \pm 2$ with equal probability. The signal is demodulated in an optimal manner and then an optimal threshold is used in making a decision. For demodulation assume the received signal $r(t)$ is mixed with a cosine wave of amplitude 1 with the same frequency and phase of the transmitted signal of interest and then low pass filtered by integrating from 0 to T. With this approach suppose at the output of the demodulator (just prior to the threshold comparator) the noise (call it n) is characterized by the Laplacian density

$$p(n) = \frac{1}{\sqrt{2}\sigma} e^{-|n|\sqrt{2}/\sigma}, \quad (-\infty < n < \infty)$$

where we will take $\sigma = 1$.

a. Using the optimal detector (with the corresponding optimal threshold) as described above compute the probability of a correct decision, P_c, for the value of A_1. You may assume that f_c is known to the receiver.

   Solution: Clearly, the optimal threshold is $T = 0$. We can assume without loss of generality that $A_1 = 2$. After mixing and applying the LPF, the density of the signal plus noise is centered at $A_1/2 = 1$. Thus,

$$\begin{aligned} P_c &= \frac{1}{\sqrt{2}} \int_0^\infty e^{-|x-1|\sqrt{2}} dx \\ &= \frac{1}{\sqrt{2}} \int_0^1 e^{(x-1)\sqrt{2}} dx + \frac{1}{\sqrt{2}} \int_1^\infty e^{-(x-1)\sqrt{2}} dx \\ &= 1 - \frac{1}{2} e^{-\sqrt{2}} = 0.88 \end{aligned}$$

b. Now suppose an interfering signal is also present so that

$$r(t) = A_1 \cos(2\pi f_c t) + A_2 \cos(2\pi f_c t + \pi/4) + n(t) \quad 0 \le t \le T.$$

   However, the receiver does not know the interfering signal is present so the receiver does not change the threshold used in part (a) nor does the receiver make any other modifications. Find the value of A_2 so that if a symbol is transmitted with $A_1 = 2$ then the probability of a correct decision for that symbol is now 0.1.

Solution: Now after mixing and low pass filtering the signal plus interferer plus noise is centered at

$$\mu = \frac{A_1}{2} + \frac{A_2}{2}\cos(\pi/4)$$
$$\mu = 1 + \frac{A_2}{2}\cos(\pi/4)$$

We then solve

$$\begin{aligned} P_c &= \frac{1}{\sqrt{2}}\int_0^\infty e^{-|x-\mu|\sqrt{2}}dx = 0.1 \\ &= \frac{1}{\sqrt{2}}\int_0^\mu e^{(x-\mu)\sqrt{2}}dx + \frac{1}{\sqrt{2}}\int_\mu^\infty e^{-(x-\mu)\sqrt{2}}dx = 0.1 \\ &= 1 - \frac{1}{2}e^{-\mu\sqrt{2}} = 0.1 \end{aligned}$$

which gives

$$\mu = -\frac{2\ln 0.9}{\sqrt{2}} = 0.149$$

and then

$$A_2 = -2.41$$

c. Using the value of A_2 found in part (b) find the probability of a correct decision at the receiver assuming $A_1 = -2$.

Solution: Now after mixing and low pass filtering the signal plus interferer plus noise is centered at

$$\mu = -1 - \frac{2.41}{2}\cos(\pi/4) = -1.851$$

We compute

$$\begin{aligned} P_c &= \frac{1}{\sqrt{2}}\int_{-\infty}^0 e^{-|x-\mu|\sqrt{2}}dx \\ &= \frac{1}{\sqrt{2}}\int_{-\infty}^\mu e^{(x-\mu)\sqrt{2}}dx + \frac{1}{\sqrt{2}}\int_\mu^0 e^{-(x-\mu)\sqrt{2}}dx \\ &= 1 - \frac{1}{2}e^{\mu\sqrt{2}} = 0.964 \end{aligned}$$

d. Combine parts (b) and (c) to compute the overall probability of a correct decision in the presence of the interferer.

Solution: With equal-likely signals we find

$$P_c = \frac{0.1 + 0.964}{2} = 0.532$$

e. Again using the value of A_2 found in part (b) find the overall probability of a correct decision if the interferer's frequency is changed such that now $f_0 = 0.8$ kHz and thus

$$r(t) = A_1 \cos(2\pi f_c t) + A_2 \cos(2\pi f_0 t + \pi/4) + n(t) \quad 0 \leq t \leq T.$$

Solution: Now after mixing and low pass filtering the signal plus interferer plus noise is centered at

$$\begin{aligned}
\mu &= \int_0^T [\frac{A_1}{2} + \frac{A_2}{2}\cos(2\pi(f_c - f_0)t + \frac{\pi}{4})]dt \\
&= \frac{A_1}{2} + \int_0^1 [\frac{A_2}{2}\cos(2\pi(f_c - f_0)t + \frac{\pi}{4})]dt \\
&= \frac{A_1}{2} + \int_0^1 [\frac{A_2}{2}\cos(400\pi t + \frac{\pi}{4})]dt \\
&= \frac{A_1}{2}
\end{aligned}$$

Thus, $P_c = 0.88$ as found in part (a).