EE 567

Homework 8 Solution

Problem 1. A/D Analysis.
A certain A/D converter has n physical bits. Suppose the input to the A/D is the waveform

$$s(t) = A\cos(2\pi f_c t) + n(t)$$

where $A\cos(2\pi f_c t)$ is the signal component and $n(t)$ is AWGN which at any time t has mean zero and variance σ^2. You can assume A and σ have units of volts.

a. Let us denote the maximum level of the A/D by M volts. We can apply a gain, G, to the input waveform to control clipping by the A/D. Find the maximum value of G so that if we sample the waveform when the signal component is at its peak value the probability of clipping is $P(clip) \leq 0.05$. You may use the fact that if Z is a standard normal random variable (mean zero, unit variance Gaussian), then $P(Z \geq z_\alpha) = \alpha$ where $z_\alpha = 1.645$ for $\alpha = 0.05$. Write your answer for G as a function of M, A and σ.

Solution: At some time $t = t_0$, the signal has the maximum value A and at that point the A/D sees $A + n(t) \sim N(A, \sigma^2)$. Applying the gain G we thus have $X = GA + Gn(t) \sim N(GA, G^2\sigma^2)$. We calculate

$$\begin{aligned} P(X \geq M) &= P\left(\frac{X - GA}{G\sigma} \geq \frac{M - GA}{G\sigma}\right) \\ &= P\left(Z \geq \frac{M - GA}{G\sigma}\right) = \alpha \end{aligned}$$

so

$$\frac{M - GA}{G\sigma} = z_\alpha$$

which yields

$$G = \frac{M}{A + z_\alpha \sigma}$$

where $z_\alpha = 1.645$.

b. Now suppose that instead of applying a gain G to control clipping we adjust the meaning of level changes within the A/D (so we set $G = 1$). Let use denote a level change by Δ volts (this is the quantization step size). For an n bit A/D converter find the largest value of Δ so that if we sample the waveform when the signal component is at its peak value the probability of clipping is $P(clip) \leq 0.05$. You should assume that an input value of zero volts lies directly in between the two middle levels of the A/D. Write your answer for Δ as a function of n, A and σ.

   Solution: We sustitute

$$M = (2^n - 1)\frac{\Delta}{2}$$

   in the expression for G found in part (a) and then setting $G = 1$ we find

$$\Delta = \frac{A + z_\alpha \sigma}{2^{n-1} - \frac{1}{2}}$$

   where $z_\alpha = 1.645$.

Problem 2. In class we derived the probability of bit error for BPSK modulation as

$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right).$$

Plot this expression for E_b/N_0 ranging from -3 to 16 dB. Your y-axis should be on a log scale.

Solution: Plot the probability of bit error for BPSK in MATLAB,

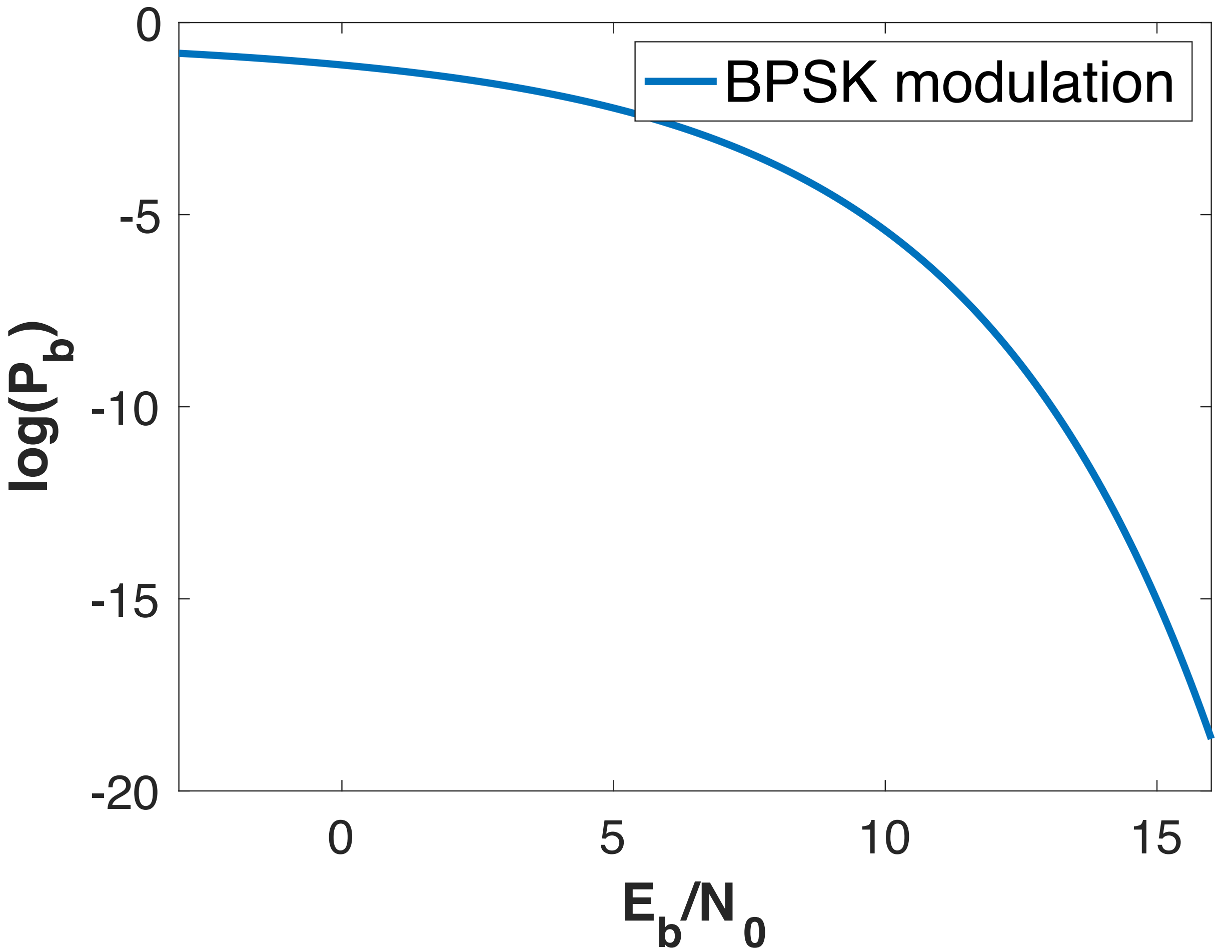

Figure 1: Probability of bit error for BPSK modulation

Problem 3. BPSK Signaling.
Suppose we receive a BPSK signal of the form

$$r(t) = A\cos(2\pi f_c t + \theta) + n(t), \quad 0 \le t \le T$$

where $n(t)$ is an AWGN process and $A = \pm 1$ equally likely. The signal is demodulated in an optimal manner and then an optimal threshold is used in making a decision. For demodulation assume the received signal $r(t)$ is mixed with a cosine wave with the same frequency and phase of the transmitted signal of interest and then low pass filtered by integrating from 0 to T. This is coherent detection with a matched filter. With this approach we derived in class the probability of bit error as

$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right).$$

a. Suppose in the mixing signal the receiver does not use the exact correct phase but instead uses $\cos(2\pi f_c t + \hat{\theta})$. Define $\Delta\theta = \theta - \hat{\theta}$. What is the correct expression for P_b in this case?

Solution: We derived that for antipodal BPSK signaling with equally likely symbols

$$P_b = \int_{(a_1+a_2)/2}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-(z-a_2)^2/2\sigma_0^2} dz$$

where a_1 is the mean of the test statistic $z(T)$ resulting after coherent detection when $A = +1$, i.e., $a_1 = |A|T/2$, and a_2 is the mean when $A = -1$, , i.e., $a_2 = -|A|T/2$, σ_0^2 is the variance of the noise in the test statistic. Substituting $u = \frac{z-a_2}{\sigma_0}$ we got

$$P_b = \int_{(a_1-a_2)/2\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

or

$$P_b = Q\left(\frac{a_1 - a_2}{\sigma_0}\right)$$

where

$$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du.$$

Letting $a_1 = \sqrt{E}$, $a_2 = -\sqrt{E}$ and $\sigma_0^2 = \frac{N_0}{2}$ we obtained the expression

$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

given above. We now let $a_1 = \sqrt{E}\cos\Delta\theta$, $a_2 = -\sqrt{E}\cos\Delta\theta$ to get

$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\cos\Delta\theta\right).$$

b. Now suppose $\Delta\theta = 0$ but we have an interferer or jammer present of the form

$$J(t) = A_J \cos(2\pi f_c t + \theta)$$

where, $A_J = 1/2$ always (does not change sign). So the receiver processes $r(t)+J(t)$. But the receiver does not know the jammer is present. Derive the expression for P_b in this case?

Solution: Assuming first that symbol 2 was sent we get

$$\begin{aligned} P_{b,2} &= \int_{(a_1+a_2)/2}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-(z-\tilde{a}_2)^2/2\sigma_0^2} dz \\ &= \int_0^{\infty} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-(z-\tilde{a}_2)^2/2\sigma_0^2} dz \end{aligned}$$

where $\tilde{a}_2 = -\sqrt{E}/2$. Substituting $u = \frac{z-\tilde{a}_2}{\sigma_0}$ we get

$$P_{b,2} = \int_{-\tilde{a}_2/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} dz$$

or

$$P_{b,2} = Q\left(\sqrt{\frac{E_b}{2N_0}}\right).$$

If symbol 1 was sent we get

$$\begin{aligned} P_{b,1} &= \int_{-\infty}^{(a_1+a_2)/2} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-(z-\tilde{a}_1)^2/2\sigma_0^2} dz \\ &= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-(z-\tilde{a}_1)^2/2\sigma_0^2} dz \end{aligned}$$

where $\tilde{a}_1 = 3\sqrt{E}/2$. Substituting $u = \frac{z-\tilde{a}_1}{\sigma_0}$ we get

$$\begin{aligned} P_{b,1} &= \int_{-\infty}^{-\tilde{a}_1/\sigma_0} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} dz \\ &= \int_{\tilde{a}_1/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} dz \end{aligned}$$

or

$$P_{b,1} = Q\left(\sqrt{\frac{9E_b}{2N_0}}\right).$$

Since symbol 1 and symbol 2 are equally likely we get

$$P_b = P_{b,1} \times \frac{1}{2} + P_{b,2} \times \frac{1}{2}$$

or

$$P_b = Q\left(\sqrt{\frac{9E_b}{2N_0}}\right) \times \frac{1}{2} + Q\left(\sqrt{\frac{E_b}{2N_0}}\right) \times \frac{1}{2}.$$

c. Now say we have the ideal case again ($\Delta\theta = 0$ and no jammer) but we decide to implement a single bit error correction code, specifically, a (7,4) Hamming code. With P_b denoting the probability of an undecoded bit error, write down the mathematical expression you could use to compute the probability of an incorrect codeword after decoding (you do not have to evaluate this expression).

Solution: With P_{cw} denoting the probability of a codeword error then a codeword error occurs if we have 2 or more bits errors in a codeword so

$$P_{cw} = \sum_{k=2}^{7} \binom{n}{k} P_b^k (1 - P_b)^{7-k}.$$

Appendix: MATLAB code

Problem 2:

```
clear; clc; close all;
Eb_N0=[-3:0.1:16];
Pb=qfunc(sqrt(2*10.^(Eb_N0/10)));
plot(Eb_N0,log10(Pb),'LineWidth',3);
set(gca,'FontSize',20);
xlabel('E_b/N_0','FontWeight','Bold'); xlim([-3,16]);
ylabel('log(P_b)','FontWeight','Bold');
legend('BPSK modulation','FontSize',24);
```