EE 567

Homework 7 Solution

Problem 1. In class we had that the gain for a parabolic antenna is

$$g = \left(\frac{\pi}{3}\right)^2 \left(\frac{d \cdot f_c}{10^8}\right)^2$$

and the half-power beamwidth is

$$\phi_b = 3.06 \left(\frac{10^8}{d \cdot f_c}\right) \quad \text{radians.}$$

The actual gain of the antenna will be reduced if there is any pointing error ϕ_e and the resulting equation is

$$g(\phi_e) = g \left[\frac{2J_1(\pi d\phi_e/\lambda)}{(\pi d\phi_e/\lambda)}\right]$$

where λ is the wavelength. Using the Bessel function approximation for small argument

$$J_1(x) \approx \frac{x}{2}\left(1 - \frac{x^2}{8}\right) \approx \frac{x}{2}e^{-x^2/8}$$

we obtain

$$g(\phi_e) \approx ge^{-2.6(\phi_e/\phi_b)^2}.$$

Plot the antenna gain (in dB) versus d/λ for $\phi_e = 0,\ 0.01,\ 0.05,\ 0.1,\ 0.15$ degrees. You should plot each of these on the same graph. Vary d/λ from 10 to 3000. You should be able to deduce from your results that pointing errors prevent the use of very narrow beams.

Solution:
Using the approximation

$$J_1(x) \approx \frac{x}{2}\left(1 - \frac{x^2}{8}\right) \approx \frac{x}{2}e^{-x^2/8},$$

we can write $g(\phi_e)$ as

$$g(\phi_e) = g \cdot \left[\frac{2J_1\left(\frac{\pi d\phi_e}{\lambda}\right)}{\frac{\pi d\phi_e}{\lambda}} \right]^2$$
$$\approx g \cdot \left[\frac{\frac{\pi d\phi_e}{\lambda} e^{-\left(\frac{\pi d\phi_e}{\lambda}\right)^2/8}}{\frac{\pi d\phi_e}{\lambda}} \right]^2$$
$$= g e^{-\frac{\pi^2}{4}\left(\frac{d\phi_e}{\lambda}\right)^2}.$$

In addition, we have the half-power beamwidth

$$\phi_b = 3.06\left(\frac{10^8}{df_c}\right) = \frac{3.06}{3} \cdot \frac{\lambda}{d}.$$

Hence,

$$g(\phi_e) \approx g e^{-\frac{(1.02)^2\pi^2}{4}\left(\frac{\phi_e}{\phi_b}\right)^2} = g e^{-2.567\left(\frac{\phi_e}{\phi_b}\right)^2}.$$

The equation of antenna gain tells us that for a fixed beamwidth ϕ_b, it is an exponential squared loss function of the pointing error ϕ_e.

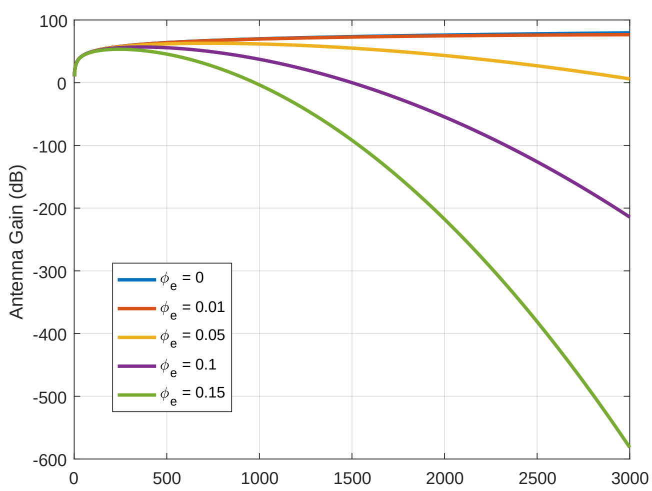

Figure 1: Parabolic antenna gain versus d/λ. Notice that the gain decreases significantly with ϕ_e if we have a narrow beam (large d/λ).

We observe that the half-power beamwidth ϕ_b is proportional to λ/d (wavelength/diameter) ratio. If we would like to use a narrow beam (small ϕ_b or large d/λ), then the pointing error ϕ_e has to be small. Otherwise, we are going to have a huge loss on antenna gain due to pointing error.

Problem 2. A signal of the form

$$s_1(t) = A\cos(2\pi f_c t + \theta)$$

is transmitted to a receiver. The signal waveform is T seconds long. A multipath version of the signal, $s_2(t)$, delayed by τ seconds also arrives at the receiver (you can think of this multipath signal as being just like $s_1(t)$ except it is delayed and the amplitude is possibly scaled). The cross-correlation of the two real signals is defined by

$$\gamma_{12} = \frac{1}{T}\int_0^T s_1(t)s_2(t)dt.$$

Determine the smallest value of τ that can be tolerated to ensure that the cross-correlation of the direct path and multipath signal is less than 5% of the direct signal power. Assume that $f_c T \gg 1$ and the multipath signal has an amplitude that is 70% that of the direct signal when received.

Solution:
Given that $s_1(t) = A\cos(2\pi f_c t + \theta)$ and $s_2(t) = \frac{7A}{10}\cos(2\pi f_c(t-\tau)+\theta)$, we have the cross-correlation of the two signal

$$\begin{aligned}\gamma_{12} &= \frac{7A^2}{10T}\int_0^T \cos(2\pi f_c t + \theta)\cos(2\pi f_c(t-\tau)+\theta)dt \\ &= \frac{7A^2}{20T}\int_0^T \cos(4\pi f_c t - 2\pi f_c \tau + 2\theta) + \cos(2\pi f_c \tau)dt \\ &\approx \frac{7A^2}{20}\cos(2\pi f_c \tau).\end{aligned}$$

Notice that the first term of the integral is approximately 0 since the integration time T satisfies $f_c T \gg 1$. Finally, we want to find the smallest value of τ such that $\gamma_{12} \leq 0.05 \times \frac{A^2}{2}$. Hence,

$$\begin{aligned}
&\frac{7A^2}{20}\cos(2\pi f_c\tau) \leq 0.05 \times \frac{A^2}{2} \\
&\Rightarrow \cos(2\pi f_c\tau) \leq 0.0714 \\
&\Rightarrow 2\pi f_c\tau \geq \cos^{-1}(0.0714)\,, \text{ since } \cos^{-1}(\cdot) \text{ is a decreasing function} \\
&\Rightarrow \tau \geq \frac{\cos^{-1}(0.0714)}{2\pi f_c}.
\end{aligned}$$

Problem 3. Use Matlab. Suppose we receive the analog signal

$$r_a(t) = A\cos(2\pi 200t + \theta)$$

and sample it at 2000 Hz to get the digital signal

$$r(n) = A\cos(0.2\pi n + \theta).$$

Suppose now we quantize the digital signal to get

$$r_q(n) = \text{Round}\left[A\cos(0.20\pi n + \theta)\right]$$

where 'Round' means the samples are rounded to the nearest integer. The amplitude A is a constant but we do not know its value. Furthermore, we do not know that the phase is $\theta = \pi/4$. We can follow the steps below to estimate the value of A. [For purposes of calculation let A actually have the value 10.]

S1. Multiply $r_q(n)$ by $x_1(n)$ and $x_2(n)$, where $x_1(n) = \cos(0.2\pi n)$ and $x_2(n) = \sin(0.2\pi n)$. Call the results $y_1(n)$ and $y_2(n)$, respectively.

S2. Simply add up the values of $y_1(n)$ and $y_2(n)$ for $n = 0, 1, 2, \ldots N-1$ (some N) and take the average of each (divide by N) and then multiply the averages by 2. Call the results z_1 and z_2, respectively.

S3. Compute $\sqrt{z_1^2 + z_2^2}$. This is the estimate of A.

a. Follow the 3 steps above and estimate A using $N = 5$.

b. Repeat (a) for $N = 10$.

c. What values of N makes your estimate of A exact if the input samples were not quantized.

d. Based on your answers to parts (a) and (b) what would your estimate for A be if N is 1000. Explain why the estimate is not becoming exact even for very large N.

e. If we change the sampling frequency to $2000 \times \pi/3$ Hz will our estimate for A become exact for large N? If so, explain why and also show that the estimate becomes exact using Matlab.

Solution:

a. If $N = 5$, then $\hat{A} = 10.1714$.

b. If $N = 10$, then $\hat{A} = 10.1714$.

c. Any N that is a multiple of 5 gives us the exact estimate of A if the input samples weren't quantized by the *round* function.

d. However, due to the fact that $r_q(n)$ is peroidic (with peroid $N = 5$),the quantization error keeps adding up when we perform summation. In order to obtain a better estimate of A, we need to make the sequence $r_q(n)$ aperiodic.

e. Since we prevent $r_q(n)$ from being periodic, the estimate of A will eventually approaches the true A as we wash out the error by increasing N.