EE 567

Homework 6 Solution

Problem 1. Let $X(t) = X(u,t)$ denote a random process described by

$$X(t) = \begin{cases} e^{-ut}, & t \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

where, u is a realization of a uniform (0,1) random variable. Define $Y(t) = Y(u,t)$ as follows:

$$Y(t) = \begin{cases} 1, & X(t) \geq e^{-2} \\ 0, & \text{elsewhere.} \end{cases}$$

Compute the correlation $R_Y(t_1, t_2)$.

Solution: We find

$$\begin{aligned} P\left(Y(t) = 1\right) &= P\left(X(t) \geq e^{-2}\right) \\ &= P\left(e^{-ut} \geq e^{-2}\right) \\ &= P(ut \leq 2) \\ &= P(u \leq 2/t) \\ &= \min\left\{\frac{2}{t}, 1\right\}. \end{aligned}$$

Now

$$R_Y(t_1, t_2) = E\left[Y(t_1)Y(t_2)\right] = P\left(Y(t_1)Y(t_2) = 1\right) = P\left(Y(t_1) = 1, Y(t_2) = 1\right).$$

Suppose $t_2 \geq t_1$. Then

$$\begin{aligned} P\left(Y(t_1) = 1, Y(t_2) = 1\right) &= P\left(Y(t_1) = 1 | Y(t_2) = 1\right) P(Y(t_2) = 1) \\ &= 1 \cdot P(Y(t_2) = 1) \\ &= \min\left\{\frac{2}{t_2}, 1\right\}. \end{aligned}$$

On the other hand if $t_1 > t_2$ then

$$\begin{aligned} P\left(Y(t_2) = 1, Y(t_1) = 1\right) &= P\left(Y(t_2) = 1 | Y(t_1) = 1\right) P(Y(t_1) = 1) \\ &= 1 \cdot P(Y(t_1) = 1) \\ &= \min\left\{\frac{2}{t_1}, 1\right\}. \end{aligned}$$

Putting these two cases together we see that

$$R_Y(t_1, t_2) = \min\left\{\frac{2}{\max\{t_1, t_2\}}, 1\right\}.$$

Problem 2. Let $X_k,\ k = 1, 2, \ldots, n$ be a sequence of independent and identically distributed random variables each with mean μ and variance σ^2. Let

$$S^2 = \frac{1}{n-1}\sum_{k=1}^{n}\left(X_k - \bar{X}\right)^2.$$

Show that $E\left[S^2\right] = \sigma^2$ (do not assume that X_k is Gaussian).

Solution:
This problem asks you to show that sample variance S^2 is an unbiased estimator of the population variance σ^2. It is simple given the mean and variance of the sample mean $\bar{X}$,

$$E[\bar{X}] = E\left[\frac{1}{n}\sum_{k=1}^{n} X_k\right] = \mu,$$

$$\operatorname{Var}(\bar{X}) = E[(\bar{X} - E[\bar{X}])^2] = \frac{\sigma^2}{n}.$$

The sample variance can be rewritten as,

$$\begin{aligned}
S^2 &= \frac{1}{n-1}\sum_{k=1}^{n}[(X_k - \mu) - (\bar{X} - \mu)]^2 \\
&= \frac{1}{n-1}\sum_{k=1}^{n}[(X_k - \mu)^2 - 2(X_k - \mu)(\bar{X} - \mu) + (\bar{X} - \mu)^2] \\
&= \frac{1}{n-1}\left[\sum_{k=1}^{n}(X_k - \mu)^2 - 2\sum_{k=1}^{n}(X_k - \mu)(\bar{X} - \mu) + \sum_{k=1}^{n}(\bar{X} - \mu)^2\right] \\
&= \frac{1}{n-1}\left[\sum_{k=1}^{n}(X_k - \mu)^2 - 2n(\bar{X} - \mu)^2 + n(\bar{X} - \mu)^2\right] \\
&= \frac{1}{n-1}\left[\sum_{k=1}^{n}(X_k - \mu)^2 - n(\bar{X} - \mu)^2\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
E[S^2] &= \frac{1}{n-1}\sum_{k=1}^{n} E[(X_k-\mu)^2] - \frac{n}{n-1}E[(\bar{X}-\mu)^2] \\
&= \frac{n}{n-1}\sigma^2 - \frac{n}{n-1}\mathrm{Var}(\bar{X}) \\
&= \frac{n}{n-1}\sigma^2 - \frac{1}{n-1}\sigma^2 = \sigma^2.
\end{aligned}$$

Problem 3. Suppose $X_1, X_2, \dots$ are each independent and normally distributed with mean zero and variance one (standard normal). Define $Y_1 = X_1$ and for $n = 2, 3, 4, \dots$ let

$$Y_n = \alpha X_n + (1-\alpha)Y_{n-1}, \quad 0 < \alpha < 1.$$

a. Compute $E[Y_n Y_m]$ as a closed form function of n, m and α for $n \geq m$.

b. Evaluate your expression for $E[Y_n Y_m]$ using $n = 10$, $m = 5$ and $\alpha = 0.2$.

Solution:

a. Note that Y_n is essentially a linear combination of independent Gaussian random variables, so it is also Gaussian distributed. Knowing that a Gaussian random variable is fully characterized by its mean and variance, we can thus determine the distribution of Y_n.

$$\begin{aligned}
&Y_1 = X_1 \sim \mathcal{N}(0,1), \\
&Y_2 = \alpha X_2 + (1-\alpha)Y_1 \sim \mathcal{N}(0, \alpha^2 + (1-\alpha)^2), \\
&Y_3 = \alpha X_3 + (1-\alpha)Y_2 \sim \mathcal{N}(0, \alpha^2 + \alpha^2(1-\alpha)^2 + (1-\alpha)^4), \\
&\quad\vdots \\
&Y_n \sim \mathcal{N}(0, \alpha^2[1 + (1-\alpha)^2 + \cdots + (1-\alpha)^{2(m-2)}] + (1-\alpha)^{2(m-1)}), \forall n \geq 2.
\end{aligned}$$

Then, we represent Y_n with $X_n, X_{n-1}, \cdots, X_{m+1}$, and Y_m as

$$\begin{aligned}
Y_n &= \alpha X_n + (1-\alpha)Y_{n-1} \\
&= \alpha X_n + (1-\alpha)[\alpha X_{n-1} + (1-\alpha)Y_{n-2}] \\
&= \alpha X_n + \alpha(1-\alpha)X_{n-1} + (1-\alpha)^2[\alpha X_{n-2} + (1-\alpha)Y_{n-3}] \\
&\quad\vdots \\
&= \alpha[X_n + (1-\alpha)X_{n-1} + (1-\alpha)^2 X_{n-2} + \cdots + (1-\alpha)^{n-m-1}X_{m+1}] + (1-\alpha)^{n-m}Y_m
\end{aligned}$$

Since $X_n, X_{n-1}, \cdots, X_{m+1}$ and Y_m are independent if $n \geq m$,

$$\begin{aligned}
E[Y_n Y_m] &= E[(1-\alpha)^{n-m} Y_m^2] \\
&= (1-\alpha)^{n-m} E[Y_m^2] \\
&= (1-\alpha)^{n-m} \left(\mathrm{Var}(Y_m) + E[Y_m]^2\right), \text{ where } E[Y_m] = 0 \\
&= (1-\alpha)^{n-m} \left[\alpha^2 \cdot \frac{1-(1-\alpha)^{2(m-1)}}{1-(1-\alpha)^2} + (1-\alpha)^{2(m-1)}\right] \\
&= \alpha^2 (1-\alpha)^{n-m} \frac{1-(1-\alpha)^{2(m-1)}}{1-(1-\alpha)^2} + (1-\alpha)^{n+m-2}.
\end{aligned}$$

b. Substituting $n = 10, m = 5$ and $\alpha = 0.2$, we get

$$E[Y_n Y_m] = 0.0853.$$

Problem 4. Consider a real Gaussian random sequence $x(n)$, n an integer, with

$$E[x(n)] = 0, \quad E\left[x(n)^2\right] = 1, \quad E[x(n)x(m)] = \rho^{|n-m|}$$

where $0 < \rho < 1$. Let

$$y(n) = 2x(n) + 2.$$

a. Is $x(n)$ wide sense stationary?

b. Find the covariance of $y(n)$ and state whether or not it is wide sense stationary.

Note: WSS has the same meaning here using the index n as in the lecture notes where we used t but now the "time" is discrete.

Solution:

a. A random sequence (or process) is wide sense stationary if it has constant mean and a crosscorrelation which is a function of index difference, $n - m$ (or time difference $t - \tau$ for process). Therefore, $x(n)$ is wide sense stationary.

b. y(n) is also a wide sense stationary random sequence since

$$\begin{aligned}
E[y(n)] &= 2, \\
E[y^2(n)] &= E[4x^2(n) + 8x(n) + 4] = 8, \\
E[y(n)y(m)] &= E[4x(n)x(m) + 4x(n) + 4x(m) + 4] = 4\rho^{|n-m|} + 4.
\end{aligned}$$