EE 567

Homework 3 Solution

Problem 1. You are given the baseband signal $m(t) = \cos(1000\pi t)$.

a. Sketch the spectrum of $m(t)$.

b. Sketch the spectrum of the DSB-SC signal $m(t)\cos(10,000\pi t)$.

Solution:

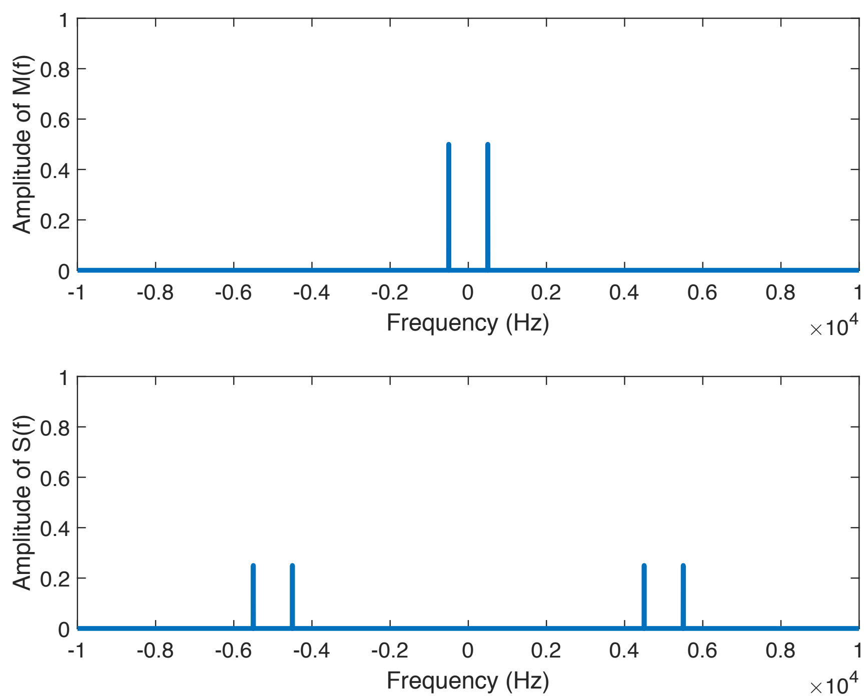

Figure 1: Spectrum of the message signal $m(t)$ and the modulated signal $s(t) = m(t)\cos(10,000\pi t)$

Problem 2. You are asked to design a DSB-SC to generate a modulated signal $s(t) = km(t)\cos(2\pi f_c t)$ where $m(t)$ is a signal band-limited to B Hz. One could simply multiply $km(t)$ by $\cos(2\pi f_c t)$ to accomplish this. However, suppose that we do not have available an oscillator to generate the signal $\cos(2\pi f_c t)$ but we do have available an oscillator that can generate the signal

$\cos^3(2\pi f_c t)$. We also have a bandpass filter that we can tune to any frequency and bandwidth. Can we still generate $s(t)$ with this equipment? If so, explain how. Hint: Make use of a trig identity involving $\cos^3(\theta)$.

Solution: Using the trigonometric identity,

$$\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta),$$

we can generate a carrier signal $\cos(2\pi f_c t)$ with the availability of $\cos^3(2\pi f_c t)$ and a lowpass(or bandpass) filter since

$$\cos^3(2\pi f_c t) = \frac{1}{4}\cos(2\pi \cdot 3f_c t) + \frac{3}{4}\cos(2\pi f_c t).$$

Problem 3. In a DSB-SC amplitude modulation system the message signal is $m(t) = e^{-t}u(t-2)$ and the carrier signal is $\cos(2000\pi t)$.

a. Find the Fourier transform of the message signal and plot its magnitude.

b. Find the Fourier transform of the modulated signal and plot its magnitude.

Solution:

a. Calculate the Fourier transform of $m(t)$ directly as

$$\begin{aligned} M(f) &= \int_{-\infty}^{\infty} m(t)e^{-j2\pi ft}dt \\ &= \int_{2}^{\infty} e^{(-1-j2\pi f)t}dt \\ &= \frac{1}{-1-j2\pi f}e^{(-1-j2\pi f)t}\Big|_{t=2}^{\infty} \\ &= \frac{e^{-2-j4\pi f}}{1+j2\pi f}. \end{aligned}$$

b. Calculate the Fourier transform of $s(t) = m(t)cos(2000\pi t)$ directly as

$$S(f) = \frac{1}{2}M(f-1000) + \frac{1}{2}M(f+1000)$$

Plot the Fourier transform of the message signal $m(t)$ and the modulated signal $s(t)$ as

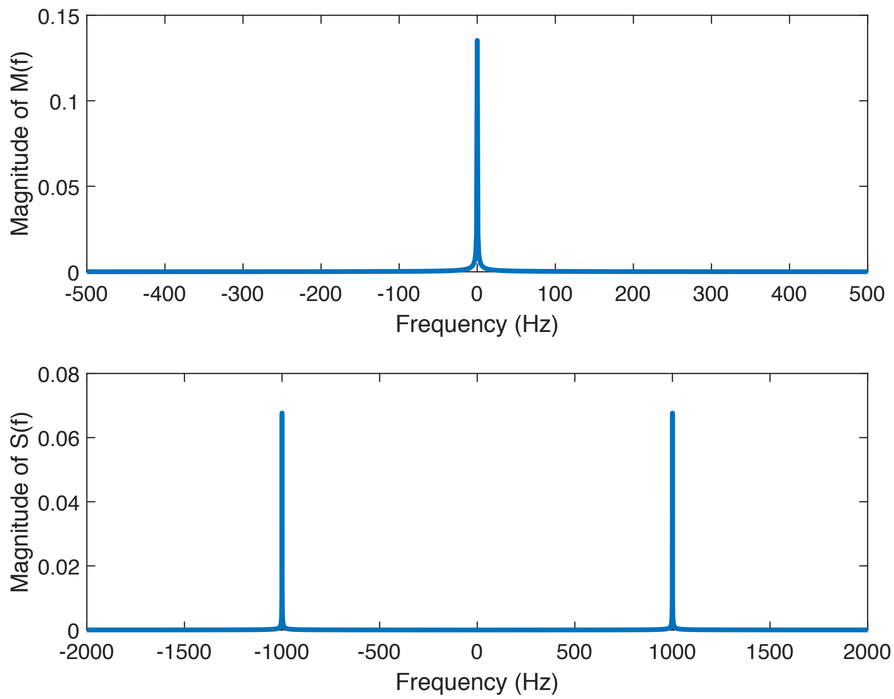

Figure 2: Spectrum of the message signal $m(t)$ and the modulated signal $s(t) = m(t)\cos(2000\pi t)$

Problem 4. In this problem you are to use a software tool (such as Matlab). Suppose you transmit to someone the message signal

$$m(t) = \begin{cases} 1000t, & 0 \leq t \leq 1/2 \text{ msec}, \\ 1 - 1000t, & 1/2 < t \leq 1 \text{ msec}. \end{cases}$$

The actual transmission is accomplished by using DSB-SC via

$$s(t) = m(t)\cos(2\pi f_c t)$$

where, $f_c = 1$ MHz. This is a finite duration signal so it has an infinite nonzero frequency content. At the receiver a bandpass filter is applied to

$s(t)$ so necessarily $m(t)$ cannot be recovered perfectly in an actual system. Suppose an ideal bandpass filter $H_B(f)$ is used with height unity centered at $\pm f_c$ that extends from $f_c - B$ to $f_c + B$ and $-f_c - B$ to $-f_c + B$.

Let us define a distortion measure as

$$D_B = 10 \cdot \log_{10} \frac{\int_{-\infty}^{\infty} |S(f)H_B(f)|^2 df}{\int_{-\infty}^{\infty} |S(f)|^2 df} \quad \text{(dB)}$$

so that no distortion ($B = \infty$) corresponds to $D_B = 0$ dB.

a. Plot D_B for $B = 10$ KHz down to $B = 100$ Hz.

b. Determine the smallest value of B such that $D_B > -2$ dB.

Solution:

a. Notice that the amplified and time-shifted message signal $m'(t) = 2 \cdot m(t + 0.0005)$ is equivalent to a triangular pulse trig($2000t$), where

$$\text{trig}(t) = \begin{cases} 1 + t, & \text{if } t \in [-1, 0] \\ 1 - t, & \text{if } t \in (0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

We have the Fourier transform pair $\text{trig}(t) \leftrightarrow \text{sinc}^2(f)$ and thus $m'(t) = \text{trig}(2000t) \leftrightarrow \frac{1}{2000}\text{sinc}^2(\frac{f}{2000})$. Using the relationship mentioned above, we have

$$|M(f)|^2 = \frac{|M'(f)|^2}{4} = \frac{\text{sinc}^4(f/2000)}{4 \cdot 4 \cdot 10^6}.$$

With the assumption that an ideal bandpass filter $H_B(f)$ applied on the modulated signal $s(t)$, the distortion measure can be simplified further as

$$\begin{aligned} D &= 10 \log \left(\frac{\int_{-\infty}^{\infty} |S(f)H_B(f)|^2 df}{\int_{-\infty}^{\infty} |S(f)|^2 df} \right) \\ &= 10 \log \left(\frac{\int_{-B}^{B} |M(f)|^2 df}{\int_{-\infty}^{\infty} |M(f)|^2 df} \right) \\ &= 10 \log \left(12000 \int_{-B}^{B} |M(f)|^2 df \right), \end{aligned}$$

since the Parseval's theorem gives $\int_{-\infty}^{\infty} |M(f)|^2 df = \int_{-\infty}^{\infty} |m(t)|^2 dt = \frac{1}{12000}$.

a. Plot D_B v.s. B as

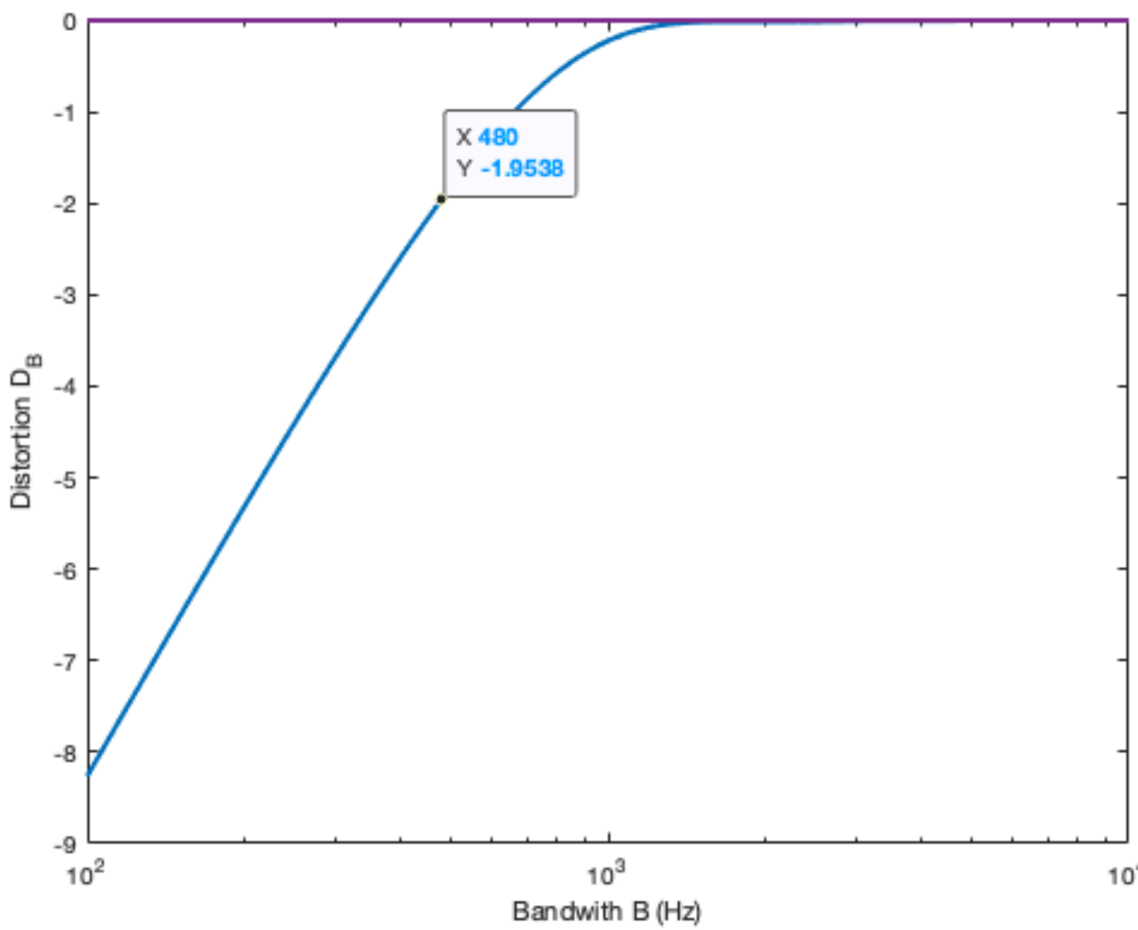

Figure 3: Distortion D_B measure v.s. bandwidth of bandpass filter $H_B(f)$

b. As shown in Figure 3, the smallest value of B that yields $D_B >$ -2 dB is around 480 Hz.

Appendix: MATLAB code

Problem 1:

```
clc; clear; close all;
Fs=20000;
T=1/Fs;
L=5000;
t=(0:L-1)'*T;
f=Fs*(-L/2:L/2-1)'/L;
m=cos(1000*pi*t);
M=fftshift(fft(m))/L;
figure;
subplot(2,1,1); plot(f,abs(M),'linewidth',2);
xlabel('Frequency (Hz)'); ylabel('Amplitude of M(f)'); ylim([0,1]);
s=m.*cos(10000*pi*t);
S=fftshift(fft(s))/L;
subplot(2,1,2); plot(f,abs(S),'linewidth',2);
xlabel('Frequency (Hz)'); ylabel('Amplitude of S(f)'); ylim([0,1]);
```

Problem 3:

```
clc; clear; close all;
f=-500:1:500;
Mf=exp(-2*(1+1j*2*pi*f))./(1+1j*2*pi*f);
figure;
subplot(2,1,1); plot(f,abs(Mf),'linewidth',2);
xlabel('Frequency (Hz)'); ylabel('Magnitude of M(f)');
f=-2000:1:2000;
Sf=0.5*exp(-2*(1+1j*2*pi*(f-1000)))./(1+1j*2*pi*(f-1000))...
   +0.5*exp(-2*(1+1j*2*pi*(f+1000)))./(1+1j*2*pi*(f+1000));
subplot(2,1,2), plot(f,abs(Sf),'linewidth',2);
xlabel('Frequency (Hz)'); ylabel('Magnitude of S(f)');
```

Problem 4:

```
clc; clear; close all;
B=[100:10:10000]';
M=@(f)sinc(f/2000).^4/(4*4*1e6);
D=zeros(length(B));
for i=1:length(B)
   D(i)=10*log10(12000*integral(M,-B(i),B(i)));
end
semilogx(B,D,'linewidth',2);
xlabel('Bandwith B (Hz)'); ylabel('Distortion D_B');
```