EE 567 Final Exam Part 1 Solution

December 17, 2019

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Problem	Points	Score
1	10	
2	12	
3	16	
4	12	
Part 1	50	

Instructions and Information:

1) Print your name and indicate on-campus or DEN student at the top of the page.

2) Make sure this part of the exam has 4 problems.

3) This is a closed book exam. You may use two 8.5x11 inch sheet of notes (front and back). You may use a self-contained calculator but not a computer. Cell phones or any device with internet capability is not permitted. **You have 55 minutes to take this part of the exam**.

4) The points for each problem is shown above.

Problem 1. A receiver front end has a noise figure of 10 dB and a gain of 50 dB and a bandwidth of 8 MHz. The input signal power is 10^{-11} W. The antenna temperature is 175 K. Find T_e, T_s, N_{out}, SNR_{in} and SNR_{out}. You may use $T_0 = 290$ K and Boltzmann's constant equals 1.38×10^{-23} J/K.

Solution: We find with F =noise figure, T_a = antenna temperature, G = gain, B_n = bandwidth, S_{in} = input power

$$\begin{aligned}
T_e &= (10^{F/10} - 1)T_0 = 2610\ K \\
T_s &= T_a + T_e = 2785\ K \\
N_{in} &= kT_aB_n = 1.93 \times 10^{-14} \\
N_{out} &= kT_sB_n \times 10^{G/10} = 3.07 \times 10^{-8} \\
SNR_{in} &= \frac{S_{in}}{N_{in}} = 518 = 27.1\ dB \\
S_{out} &= S_{in} \times 10^{G/10} = 1.00 \times 10^{-6} \\
SNR_{out} &= \frac{S_{out}}{N_{out}} = 32.5 = 15.1\ dB.
\end{aligned}$$

Problem 2. Suppose that BPSK modulation is used for transmitting information over an AWGN channel with a power spectral density of

$$\frac{1}{2}N_0 = 10^{-10}\ W/Hz.$$

The transmitted signal energy is

$$E_b = \frac{1}{2}A^2T.$$

where T is the bit interval and A is the signal amplitude. Determine the signal amplitude required to achieve an error probability of 10^{-6} when the data rate is

a. 1 Mbps.

Solution:

$$\begin{aligned} P_b &= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{A^2T}{N_0}}\right) = Q\left(\sqrt{\frac{A^2}{N_0R}}\right) \\ &= Q\left(\sqrt{\frac{A^2}{2\times 10^{-10}R}}\right) \Rightarrow A = Q^{-1}(10^{-6})\sqrt{2\times 10^{-10}R}. \end{aligned}$$

Thus,

$$\begin{aligned} A &= Q^{-1}(10^{-6})\sqrt{2\times 10^{-10}\cdot 10^6} \\ &= Q^{-1}(10^{-6})\times 0.0141 \quad [= 0.0814]. \end{aligned}$$

b. 10 Mbps.

Solution:

$$\begin{aligned} A &= Q^{-1}(10^{-6})\sqrt{2\times 10^{-10}\cdot 10\times 10^6} \\ &= Q^{-1}(10^{-6})\times 0.0447 \quad [= 0.2573]. \end{aligned}$$

c. 100 Mbps.

Solution:

$$\begin{aligned} A &= Q^{-1}(10^{-6})\sqrt{2\times 10^{-10}\cdot 100\times 10^6} \\ &= Q^{-1}(10^{-6})\times 0.1414 \quad [= 0.8137]. \end{aligned}$$

Problem 3. [16 pts]. Consider a signal detector with an input

$$r = \pm A + n$$

where $+A$ and $-A$ occur with equal probability and the noise n is characterized by the density

$$p(n) = \begin{cases} \frac{1}{\sqrt{2}\sigma} e^{-\sqrt{2}|n|/\sigma}, & -\infty < n < \infty \\ 0 \text{ elsewhere.} \end{cases}$$

a. Determine the probability of error formula as a function of A and σ.

Solution: Since $+A$ and $-A$ occur with equal probability we can assume WLOG that $+A$ is transmitted. Then the probability of error is

$$\begin{aligned} P_b &= \int_{-\infty}^{0} \frac{1}{\sqrt{2}\sigma} e^{-\sqrt{2}|r-A|/\sigma} dr \\ &= \int_{-\infty}^{-A} \frac{1}{\sqrt{2}\sigma} e^{\sqrt{2}u/\sigma} du \\ &= \frac{1}{2} e^{-\sqrt{2}A/\sigma}. \end{aligned}$$

b. Find the SNR ($= A^2/\sigma^2$) required to achieve an error probability of 10^{-5}.

Solution: We solve

$$P_b = \frac{1}{2} e^{-\sqrt{2}A/\sigma}$$

to get

$$\frac{A}{\sigma} = -\frac{ln(2P_b)}{\sqrt{2}} = 7.65$$

so

$$SNR = \left(\frac{A}{\sigma}\right)^2 = 58.52 = 17.67 \text{ dB}.$$

Problem 4. Suppose $X_1, X_2, \ldots$ are each independent and normally distributed with mean zero and variance one (standard normal). Define $Y_{-1} = 0$ and for $n = 0, 1, 2, \ldots$ let

$$Y_n = (1-\alpha)X_n + \alpha Y_{n-1}, \quad 0 < \alpha < 1.$$

a. Compute $E[Y_n Y_m]$ as a closed form function of n, m and α for $n \geq m$.

Solution:

$$\begin{aligned} Y_0 &= (1-\alpha)X_0 \\ Y_1 &= (1-\alpha)X_1 + \alpha(1-\alpha)X_0 \\ &\vdots \\ Y_n &= (1-\alpha)X_n + \alpha(1-\alpha)X_{n-1} + \alpha^2(1-\alpha)X_{n-2} \\ &\quad + \cdots + \alpha^{n-1}X_1 + \alpha^n(1-\alpha)X_0 \end{aligned}$$

which may be written as

$$Y_n = (1-\alpha)\sum_{k=0}^{n} \alpha^k X_{n-k}.$$

Similarly,

$$Y_m = (1-\alpha)\sum_{k=0}^{m} \alpha^k X_{m-k}.$$

Since $E[X_n X_m] = 0$ for $n \neq m$ and $E[X_n^2] = 1$ we require $j = k - n + m$ and find for $n \geq m$

$$\begin{aligned} E[Y_n Y_m] &= (1-\alpha)^2 \alpha^{-n+m} \sum_{k=n-m}^{n} \alpha^{2k} \\ &= \frac{1-\alpha}{1+\alpha}\left(\alpha^{n-m} - \alpha^{n+m+2}\right). \end{aligned}$$

b. Evaluate your expression for $E[Y_n Y_m]$ using $n = 10$, $m = 5$ and $\alpha = 0.9$.

Solution: Evaluating this expression for $n = 10,\ m = 5$ and $\alpha = 0.9$ we get

$$E[Y_{10} Y_5] = 2.23 \times 10^{-2}.$$

c. One way to implement a smoothing or low pass filter is simply to take the average of N samples as

$$Z_n = \frac{1}{N} \sum_{k=n-N+1}^{n} X_k.$$

The filter that computes Y_n in part (a) can also act like a low pass filter. Find the value of α as a function of N so that the expected value and variance of Y_n is the same as the expected value and variance, respectively, of Z_n as $n \to \infty$ for our mean zero unit variance Gaussian input. You must derive your answer mathematically for credit.

Solution: We find

$$E[Z_n] = 0, \quad Var[Z_n] = \frac{1}{N}$$

for all n. Now

$$E[Y_n] = (1-\alpha) \sum_{k=0}^{n} \alpha^k E[X_{n-k}] = 0$$

and

$$\begin{aligned} Var[Y_n] &= (1-\alpha)^2 \sum_{k=0}^{n} \alpha^{2k} Var[X_{n-k}] \\ &= \frac{1-\alpha}{1+\alpha} \left(1 - \alpha^{2(n+1)}\right) \\ &\to \frac{1-\alpha}{1+\alpha} \end{aligned}$$

as $n \to \infty$. So we solve

$$\frac{1-\alpha}{1+\alpha} = \frac{1}{N}$$

to get

$$\alpha = \frac{N-1}{N+1}.$$