EE 567 Midterm Solution

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Problem	Points	Score
1	9	
2	8	
3	8	
4	9	
5	10	
6	15	
7	15	
8	10	
9	16	
Total	100	

Instructions and Information:

1) Print your name and assigned class number at the top of the page.

2) Make sure your exam has 9 problems.

3) This is a closed book exam. You may use one 8.5x11 inch sheet of notes (front and back). You may use a self-contained calculator but not a computer. Cell phones or any device with internet capability is not permitted. **You have 1 hour and 50 minutes to take this exam**.

4) The points for each problem is shown above.

Problem 1. Determine which of the following systems is linear. You may write any or all of your answers to this problem without proof if you wish to do so.

a. $y(t) = x(t) + 2$. **Solution:** Not linear.

b. $y(t) = \cos(x(t))$. **Solution:** Not linear.

c. $y(t) = 2x(t-1)$. **Solution:** Linear.

Problem 2. An angle modulated signal is described by

$$s(t) = 10\cos\left(2\pi f_c t + 0.2\sin(2\pi f_1 t)\right)$$

where $f_c = 1$ MHz and $f_1 = 2$ kHz.

a. Find the power of the modulated signal $s(t)$.
Solution:

$$P = \frac{10^2}{2} = 50 \text{ W}.$$

b. Find the frequency deviation, Δf.
Solution:

$$\theta(t) = 2\pi f_c t + 0.25\sin(2\pi f_c t)$$

$$f_i(t) = \frac{1}{2\pi}\frac{d\theta(t)}{dt} = f_c + 0.2 f_1 \cos(2\pi f_1 t)$$

$$\Delta f = 400 \text{ Hz}.$$

Problem 3. A carrier wave of frequency 100 MHz is frequency-modulated by a sine-wave of amplitude 10 volts and frequency 25 kHz. The frequency sensitivity of the modulator is 10 kHz per volt. Determine the approximate bandwidth of the FM wave using Carson's rule.

Solution:

$$m(t) = 10\cos(2\pi f_m t), \;\; f_m = 25 \text{ kHz}$$

$$f_i(t) = f_c + k_f m(t), \;\; f_c = 100 \text{ MHz}, \;\; k_f = 10 \text{ kHz/V}$$

$$\begin{aligned}\theta(t) &= 2\pi f_c t + 2\pi k_f \int_0^t m(s)ds \\ &= 2\pi f_c t + \frac{10k_f}{f_m}\sin(2\pi f_m t)\end{aligned}$$

$$s(t) = A_c \cos\left(2\pi f_c t + \frac{10k_f}{f_m}\sin(2\pi f_m t\right)$$

$$\beta = \frac{10k_f}{f_m} \quad \text{(modulation index)}$$

$$\begin{aligned}B_{FM} &= 2f_m(1+\beta) \\ &= 50 \text{ kHz}\left(1 + \frac{10(10 \text{ kHz})}{25 \text{ kHz}}\right) \\ &= = 50 \text{ kHz} + 200 \text{ kHz} \\ &= 250 \text{ kHz}.\end{aligned}$$

Problem 4. A certain AM signal is given as

$$s_{AM}(t) = [4 + \cos(2\pi f_m t)] \cos(2\pi f_c t).$$

The value of f_c is much greater than the bandwidth of the signal.

a. What is the modulating signal, $m(t)$?
Solution:

$$m(t) = \cos(2\pi f_m t).$$

b. What is the modulation index?
Solution:

$$\mu = \frac{m_p}{A} = \frac{1}{4}.$$

c. Determine the average message power, that is, the power in $m(t)$.
Solution:

$$P = \frac{1}{2}.$$

Problem 5. Consider the random process

$$X(t) = A_1 \cos(\omega_c t + \theta) + A_2 \cos(\omega_c t + \phi)$$

where ω_c is a constant, A_1 and A_2 are continuous random variables each uniformly distributed on the interval $[-1,\ 1]$, θ and ϕ are each continuous random variables uniformly distributed on the interval $[0,\ 2\pi)$. A_1, A_2, θ and ϕ are all independent of each other.

a. Find the mean function $\mu_X(t) = E[X(t)]$.

b. Find the correlation function $R_X(t_1, t_2)$ and determine if the process is wide sense stationary.

Solution:
Since $A_1, A_2 \sim \text{Uniform}([-1,1])$ and $\theta, \phi \sim \text{Uniform}([0,2\pi))$, we have $E[A_1] = E[A_2] = 0$, $Var(A_1) = Var(A_2) = 1/3$, $E[\theta] = E[\phi] = \pi$ and $Var(\theta) = Var(\phi) = \pi^2/3$.

a.

$$\begin{aligned}
\mu_X(t) &= E[A_1 \cos(\omega_c t + \theta) + A_2 \cos(\omega_c t + \phi)] \\
&= E[A_1]E[\cos(\omega_c t + \theta)] + E[A_2]E[\cos(\omega_c t + \phi)], \text{ from independence} \\
&= 0.
\end{aligned}$$

b.

$$\begin{aligned}
R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \\
&= E[A_1^2 \cos(\omega_c t_1 + \theta)\cos(\omega_c t_2 + \theta)] + E[A_1 A_2 \cos(\omega_c t_1 + \theta)\cos(\omega_c t_2 + \phi)] \\
&\quad + E[A_1 A_2 \cos(\omega_c t_2 + \theta)\cos(\omega_c t_1 + \phi)] + E[A_2^2 \cos(\omega_c t_1 + \phi)\cos(\omega_c t_2 + \phi)] \\
&= \frac{E[A_1^2]}{2} E[\cos(\omega_c(t_1 + t_2) + 2\theta) + \cos(\omega_c(t_1 - t_2))] + 0 + 0 \\
&\quad + \frac{E[A_2^2]}{2} E[\cos(\omega_c(t_1 + t_2) + 2\phi) + \cos(\omega_c(t_1 - t_2))] \\
&= \frac{E[A_1^2]}{2} \cos(\omega_c(t_1 - t_2)) + \frac{E[A_2^2]}{2} \cos(\omega_c(t_1 - t_2)) \\
&= \frac{1}{3} \cos(\omega_c(t_1 - t_2)).
\end{aligned}$$

Hence, $X(t)$ is a wide sense stationary random process.

Problem 6. We wish to encode a signal. The signal ranges between the values -6 and +6. There are 3 bits or 8 levels (hence 8 code numbers) available. The levels assigned have symmetry like we demonstrated in class. The first three sample values obtained (before quantization) are 5.2, 0.8, and -4.3, respectively.

a. Find the quantized values for the three sample values.
   Solution: We divide the range 6 to -6 into 8 segments each of length 1.5. Then,
$$5.2 \to 5.25,\ \ 0.8 \to 0.75,\ \ -4.3 \to -3.75.$$

b. Find the corresponding Pulse Coded Modulation (PCM) sequence for the quantized values.
   Solution: 111 100 001

c. Draw the corresponding Manchester encoded waveform for the quantized values. Comment on why one may wish to utilize Manchester encoding in applications.
   Solution: See class notes where a similar example was worked.

Problem 7. In class we considered an ideal (no noise) reception of a BPSK signal

$$r(t) = A\cos(2\pi f_c t)$$

where A has the same sign for T seconds. We showed how to process this signal to determine the information (i.e., the sign of A) by mixing this signal with a locally generated signal and then integrating for T seconds (lowpass filtering) before making our decision. Suppose at the receiver we do not know the exact value of f_c (we do know the phase is 0) so we use $f_c + \Delta f$ for the frequency in the local mixing signal. Hence, Δf represents a frequency error. If $T = 1$ second find the largest value of Δf so that the power in the signal after lowpass filtering is not degraded more than 1 dB relative to the case when $\Delta f = 0$.

You can write your answer as a function of the sinc function where $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$.

Note: You may assume that $f_c \gg 1/T$ so that after lowpass filtering any terms containing sinusoids with high frequencies may be ignored.
Solution:

Ignoring double frequency terms we get after mixing and the LPF

$$\int_0^T \frac{A}{2}\cos(2\pi\Delta f t) = \frac{AT}{2}\mathrm{sinc}(2\pi\Delta f T).$$

In the ideal case with $\Delta f = 0$ we would obtain $\frac{AT}{2}$. So, upon computing power we solve

$$\mathrm{sinc}^2(2\pi\Delta f T = x$$

where $x = 0.794$ (corresponding to -1 dB) and then with $T = 1$ we thus get

$$\Delta f = \frac{\mathrm{sinc}^{-1}\sqrt{x}}{2}.$$

Problem 8. In class we considered the linearized model of a PLL via

$$\frac{d\phi_e(t)}{dt} + 2\pi k_0 \int_{-\infty}^{\infty} \phi_e(\tau)h(t-\tau)d\tau = \frac{d\phi_1(t)}{dt}$$

where $\phi_1(t)$ was the input phase to be tracked and $\phi_e(t) = \phi_1(t) - \hat{\phi}_1(t)$ was the resulting phase estimate error. Suppose the transfer function of the loop filter, $H(f)$, is of the form

$$H(f) = \frac{1}{j2\pi f + \sqrt{2}}.$$

Find the closed loop transfer function, that is, find the transfer function relating the Fourier transform of the estimate of $\phi_1(t)$ to the Fourier transform of $\phi_1(t)$.

Solution:

We find

$$j2\pi f \Phi_e(f) + 2\pi k_0 \Phi_e(f) H(f) = j2\pi f \Phi_1(f)$$

and

$$\Phi_e(f) = \frac{1}{1 + l(f)} \Phi_1(f)$$

where

$$L(f) = \frac{k_0 H(f)}{jf}.$$

Now

$$\Phi_e(f) = \Phi_1(f) - \hat{\Phi}_1(f)$$

so

$$\begin{aligned} G(f) : &= \frac{\hat{\Phi}_1(f)}{\Phi_1(f)} \\ &= 1 - \frac{1}{1 + L(f)} \\ &= \frac{k_0}{-2\pi f^2 + j\sqrt{2} f + k_0}. \end{aligned}$$

Problem 9. Suppose you transmit to someone the message signal

$$m(t) = \begin{cases} \dfrac{\sin^2(t)}{t^2}, & t \neq 0, \text{ msec} \\ 1, & t = 0. \text{ msec.} \end{cases}$$

The actual transmission is accomplished by using DSB-SC via

$$s(t) = m(t)\cos(2\pi f_c t)$$

where, $f_c = 1$ MHz. At the receiver a bandpass filter is applied to $s(t)$. Suppose an ideal bandpass filter $H_B(f)$ is used with height unity centered at $\pm f_c$ that extends from $f_c - B$ to $f_c + B$ and $-f_c - B$ to $-f_c + B$.

Let us define a distortion measure as

$$D_B = 20 \cdot \log_{10} \frac{\int_{-\infty}^{\infty} |S(f)H_B(f)|df}{\int_{-\infty}^{\infty} |S(f)|df} \quad \text{(dB)}$$

so that no distortion corresponds to $D_B = 0$ dB. Determine the smallest value of B such that $D_B > -1$ dB.

Solution:

Note that we can ignore the f_c frequency shifting here and consider the problem as low pass as discussed in class. Now

$$\frac{\sin^2(t)}{t^2} = \text{sinc}^2(t/\pi)$$

which has a Fourier transform that is a triangle symmetric about 0 with value π at $f = 0$ and extends from $f = -1/\pi$ to $f = 1/\pi$. The equation of the line segment for the right side of the triangle (positive frequency) is

$$y = -\pi^2 f + \pi.$$

The area of our triangle is 1 so the denominator of our distortion measure is unity. Hence, we wish to solve

$$2\int_0^B (-\pi^2 f + \pi)df = L_0$$

for B where $L_0 = 0.89$ with L_0 corresponding to -1 dB such that

$$L_0 = 10^{(-1/20)}.$$

Solving we find

$$B = 0.213 \text{ kHz}.$$