EE 567

Homework 6 Solution

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Problem 1. Consider an ideal bandpass filter

$$H_{BP}(f) = \begin{cases} e^{-j2\pi(f-f_c)t_0}, & f_c - B \le f \le f_c + B \\ e^{-j2\pi(f+f_c)t_0}, & -f_c - B \le f \le -f_c + B \\ 0, & \text{elsewhere.} \end{cases}$$

Let $s(t) = Ar_T(t)\cos(2\pi f_c t)$ where

$$r_T(t) = \begin{cases} 1, & -T/2 \le t \le T/2 \\ 0, & \text{elsewhere.} \end{cases}$$

Determine the response of the bandpass system to the input $s(t)$. Your answer may utilize the sine integral where the sine integral is defined by

$$\text{Si}(u) = \int_0^u \frac{\sin\lambda}{\lambda}.$$

You may assume that $f_c T \gg 1$ so that $s(t)$ may be considered narrowband.

Solution:
Consider the lowpass equivalent filter

$$H_{LP}(f) = \begin{cases} e^{-j2\pi f t_0}, & \text{if } f \in [-B, B], \\ 0, & \text{otherwise.} \end{cases}$$

And hence its time domain impulse response

$$h_{LP}(t) = 2B\text{sinc}(2B(t - t_0)),$$

where the sinc function is $\text{sinc}(x) = \frac{\sin \pi x}{\pi x}$. Then, we can compute the lowpass equivalent output $y_{LP}(t)$ as

$$\begin{aligned} y_{LP}(t) &= r_T(t) * h_{LP}(t) \\ &= 2B \int_{-T/2}^{T/2} \frac{\sin(2B\pi(t-\tau-t_0))}{2B\pi(t-\tau-t_0)} d\tau \\ &= \frac{1}{\pi} \int_{2B\pi(t-T/2-t_0)}^{2B\pi(t+T/2-t_0)} \frac{\sin\lambda}{\lambda} d\lambda \\ &= \frac{1}{\pi}\Big[\text{Si}(2B\pi(t+T/2-t_0)) - \text{Si}(2B\pi(t-T/2-t_0))\Big]. \end{aligned}$$

Finally, we have the bandpass output signal $y(t)$ as

$$y(t) = \frac{A}{\pi}\Big[\mathrm{Si}(2B\pi(t+T/2-t_0)) - \mathrm{Si}(2B\pi(t-T/2-t_0))\Big]\cos(2\pi f_c t).$$

Problem 2. Let $X_k,\ k = 1, 2, \ldots, n$ be a sequence of independent and identically distributed random variables each with mean μ and variance σ^2. Let

$$S^2 = \frac{1}{n-1}\sum_{k=1}^{n}\left(X_k - \bar{X}\right)^2.$$

Show that $E\left[S^2\right] = \sigma^2$ (do not assume that X_k is Gaussian).

Solution:
This problem asks you to show that sample variance S^2 is an unbiased estimator of the population variance σ^2. It is simple given the mean and variance of the sample mean $\bar{X}$,

$$E[\bar{X}] = E\left[\frac{1}{n}\sum_{k=1}^{n} X_k\right] = \mu,$$

$$\mathrm{Var}(\bar{X}) = E[(\bar{X} - E[\bar{X}])^2] = \frac{\sigma^2}{n}.$$

The sample variance can be rewritten as,

$$\begin{aligned}
S^2 &= \frac{1}{n-1}\sum_{k=1}^{n}[(X_k-\mu) - (\bar{X}-\mu)]^2 \\
&= \frac{1}{n-1}\sum_{k=1}^{n}[(X_k-\mu)^2 - 2(X_k-\mu)(\bar{X}-\mu) + (\bar{X}-\mu)^2] \\
&= \frac{1}{n-1}\left[\sum_{k=1}^{n}(X_k-\mu)^2 - 2\sum_{k=1}^{n}(X_k-\mu)(\bar{X}-\mu) + \sum_{k=1}^{n}(\bar{X}-\mu)^2\right] \\
&= \frac{1}{n-1}\left[\sum_{k=1}^{n}(X_k-\mu)^2 - 2n(\bar{X}-\mu)^2 + n(\bar{X}-\mu)^2\right] \\
&= \frac{1}{n-1}\left[\sum_{k=1}^{n}(X_k-\mu)^2 - n(\bar{X}-\mu)^2\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
E[S^2] &= \frac{1}{n-1}\sum_{k=1}^{n} E[(X_k-\mu)^2] - \frac{n}{n-1}E[(\bar{X}-\mu)^2] \\
&= \frac{n}{n-1}\sigma^2 - \frac{n}{n-1}\mathrm{Var}(\bar{X}) \\
&= \frac{n}{n-1}\sigma^2 - \frac{1}{n-1}\sigma^2 = \sigma^2.
\end{aligned}$$

Problem 3. Suppose $X_1, X_2, \ldots$ are each independent and normally distributed with mean zero and variance one (standard normal). Define $Y_1 = X_1$ and for $n = 2, 3, 4, \ldots$ let

$$Y_n = \alpha X_n + (1-\alpha)Y_{n-1}, \quad 0 < \alpha < 1.$$

a. Compute $E[Y_n Y_m]$ as a closed form function of n, m and α for $n \geq m$.

b. Evaluate your expression for $E[Y_n Y_m]$ using $n = 10$, $m = 5$ and $\alpha = 0.2$.

Solution:

a. Note that Y_n is essentially a linear combination of independent Gaussian random variables, so it is also Gaussian distributed. Knowing that a Gaussian random variable is fully characterized by its mean and variance, we can thus determine the distribution of Y_n.

$$\begin{aligned}
&Y_1 = X_1 \sim \mathcal{N}(0,1), \\
&Y_2 = \alpha X_2 + (1-\alpha)Y_1 \sim \mathcal{N}(0, \alpha^2 + (1-\alpha)^2), \\
&Y_3 = \alpha X_3 + (1-\alpha)Y_2 \sim \mathcal{N}(0, \alpha^2 + \alpha^2(1-\alpha)^2 + (1-\alpha)^4), \\
&\quad\vdots \\
&Y_n \sim \mathcal{N}(0, \alpha^2[1 + (1-\alpha)^2 + \cdots + (1-\alpha)^{2(m-2)}] + (1-\alpha)^{2(m-1)}), \forall n \geq 2.
\end{aligned}$$

Then, we represent Y_n with $X_n, X_{n-1}, \cdots, X_{m+1}$, and Y_m as

$$\begin{aligned}
Y_n &= \alpha X_n + (1-\alpha)Y_{n-1} \\
&= \alpha X_n + (1-\alpha)[\alpha X_{n-1} + (1-\alpha)Y_{n-2}] \\
&= \alpha X_n + \alpha(1-\alpha)X_{n-1} + (1-\alpha)^2[\alpha X_{n-2} + (1-\alpha)Y_{n-3}] \\
&\quad\vdots \\
&= \alpha[X_n + (1-\alpha)X_{n-1} + (1-\alpha)^2 X_{n-2} + \cdots + (1-\alpha)^{n-m-1}X_{m+1}] + (1-\alpha)^{n-m}Y_m
\end{aligned}$$

Since $X_n, X_{n-1}, \cdots, X_{m+1}$ and Y_m are independent if $n \geq m$,

$$
\begin{aligned}
E[Y_n Y_m] &= E[(1-\alpha)^{n-m} Y_m^2] \\
&= (1-\alpha)^{n-m} E[Y_m^2] \\
&= (1-\alpha)^{n-m} \left(\mathrm{Var}(Y_m) + E[Y_m]^2\right), \text{ where } E[Y_m] = 0 \\
&= (1-\alpha)^{n-m} \left[\alpha^2 \cdot \frac{1-(1-\alpha)^{2(m-1)}}{1-(1-\alpha)^2} + (1-\alpha)^{2(m-1)}\right] \\
&= \alpha^2 (1-\alpha)^{n-m} \frac{1-(1-\alpha)^{2(m-1)}}{1-(1-\alpha)^2} + (1-\alpha)^{n+m-2}.
\end{aligned}
$$

b. Substituting $n = 10, m = 5$ and $\alpha = 0.2$, we get

$$E[Y_n Y_m] = 0.0853.$$

Problem 4. Consider a real Gaussian random sequence $x(n)$, n an integer, with

$$E[x(n)] = 0, \quad E\left[x(n)^2\right] = 1, \quad E[x(n)x(m)] = \rho^{|n-m|}$$

where $0 < \rho < 1$. Let

$$y(n) = 2x(n) + 2.$$

a. Is $x(n)$ wide sense stationary?

b. Find the covariance of $y(n)$ and state whether or not it is wide sense stationary.

Note: WSS has the same meaning here using the index n as in the lecture notes where we used t but now the "time" is discrete.

Solution:

a. A random sequence (or process) is wide sense stationary if it has constant mean and a crosscorrelation which is a function of index difference, $n - m$ (or time difference $t - \tau$ for process). Therefore, $x(n)$ is wide sense stationary.

b. y(n) is also a wide sense stationary random sequence since

$$
\begin{aligned}
E[y(n)] &= 2, \\
E[y^2(n)] &= E[4x^2(n) + 8x(n) + 4] = 8, \\
E[y(n)y(m)] &= E[4x(n)x(m) + 4x(n) + 4x(m) + 4] = 4\rho^{|n-m|} + 4.
\end{aligned}
$$