EE 567

Homework 5 Solution

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Problem 1. Consider a first-order PLL as described in class, i.e., the loop filter is $H(f) = 1$. Then

$$\Phi_e(f) = \frac{1}{1 + K_0/jf}\Phi_1(f).$$

We wish to investigate the loop behavior in the presence of a frequency modulated input. We have

$$m(t) = A_m \cos(2\pi f_m t)$$

with the corresponding FM wave given by

$$s(t) = A_c \sin[2\pi f_c t + \beta \sin(2\pi f_m t)]$$

where β is the modulation index.

a. Give the equation for $\phi_1(t)$.

b. Let us write
$$\phi_e(t) = \phi_{e0} \cos(2\pi f_m t + \psi).$$
Give the equations for ϕ_{e0} and ψ. In your expression for ϕ_{e0} use the variable $\Delta f = \beta f_m$.

c. Plot the phase error amplitude ϕ_{e0} normalized with respect to $\Delta f/K_0$ versus the dimenionless parameter f_m/K_0.

d. For the loop to track the frequency modulation closely we require the phase error $\phi_e(t)$ to remain within the linear region which implies $\sin[\phi_e(t)] \approx \phi_e(t)$ which is true if $\phi_e(t) \leq 0.5$ radians. Explain why this requires $\Delta f \leq 0.5K_0$.

Solution:

a. Given the modulating signal $m(t) = A_m \cos(2\pi f_m t)$, the frequency sensitivity K_f, and the FM signal $s(t) = A_c \sin[2\pi f_c t + \phi_1(t)]$, we have

$$\begin{aligned}
\phi_1(t) &= 2\pi K_f \int_0^t m(s) ds \\
&= \frac{K_f A_m}{f_m} \sin(2\pi f_m t) \\
&= \beta \sin(2\pi f_m t).
\end{aligned}$$

b. Let us write $\phi_e(t) = \phi_{e0} \cos(2\pi f_m t + \psi)$, then

$$\Phi_e(f) = \frac{\phi_{e0} \cos\psi}{2} \left[\delta(f - f_m) + \delta(f + f_m)\right] - \frac{\phi_{e0} \sin\psi}{2j} \left[\delta(f - f_m) - \delta(f + f_m)\right].$$

In addition, from the first order model of PLL

$$\Phi_e(f) = \frac{1}{1 + K_0/jf} \Phi_1(f),$$

with $\Phi_1(f) = \frac{\beta}{j2} \left[\delta\left(f - f_m\right) - \delta\left(f + f_m\right)\right]$, we have

$$\begin{aligned}
\Phi_e(f) &= \frac{1}{1 + \frac{K_0}{jf}} \frac{\beta}{j2} \left[\delta\left(f - f_m\right) - \delta\left(f + f_m\right)\right] \\
&= \frac{\beta/2}{K_0/f + j} \left[\delta\left(f - f_m\right) - \delta\left(f + f_m\right)\right] \\
&= \frac{\beta K_0/2f - j\beta/2}{(K_0/f)^2 + 1} \left[\delta\left(f - f_m\right) - \delta\left(f + f_m\right)\right].
\end{aligned}$$

We then solve the follow system of equations

$$\begin{cases}
\frac{\phi_{e0} \cos\psi}{2} &= \frac{\beta K_0}{2 f_m \left(\frac{K_0^2}{f_m^2} + 1\right)} \\
-\frac{\phi_{e0} \sin\psi}{2j} &= \frac{-j\beta}{2\left(\frac{K_0^2}{f_m^2} + 1\right)}
\end{cases}$$

$$\Rightarrow \begin{cases}
\phi_{e0} \cos\psi &= \frac{\beta K_0 f_m}{K_0^2 + f_m^2} \\
\phi_{e0} \sin\psi &= \frac{-\beta f_m^2}{K_0^2 + f_m^2}
\end{cases}$$

Thus we have

$$\tan\psi = \frac{-f_m}{K_0} \Rightarrow \psi = \tan^{-1}\left(\frac{-f_m}{K_0}\right),$$

$$\begin{cases} \cos\psi & = \frac{K_0}{\sqrt{K_0^2+f_m^2}} \\ \sin\psi & = \frac{-f_m}{\sqrt{K_0^2+f_m^2}} \end{cases} \Rightarrow \phi_{e0} = \frac{\Delta f}{\sqrt{K_0^2+f_m^2}} = \frac{\Delta f/K_0}{\sqrt{1+(f_m/K_0)^2}}.$$

c. Please see Figure 1 below.

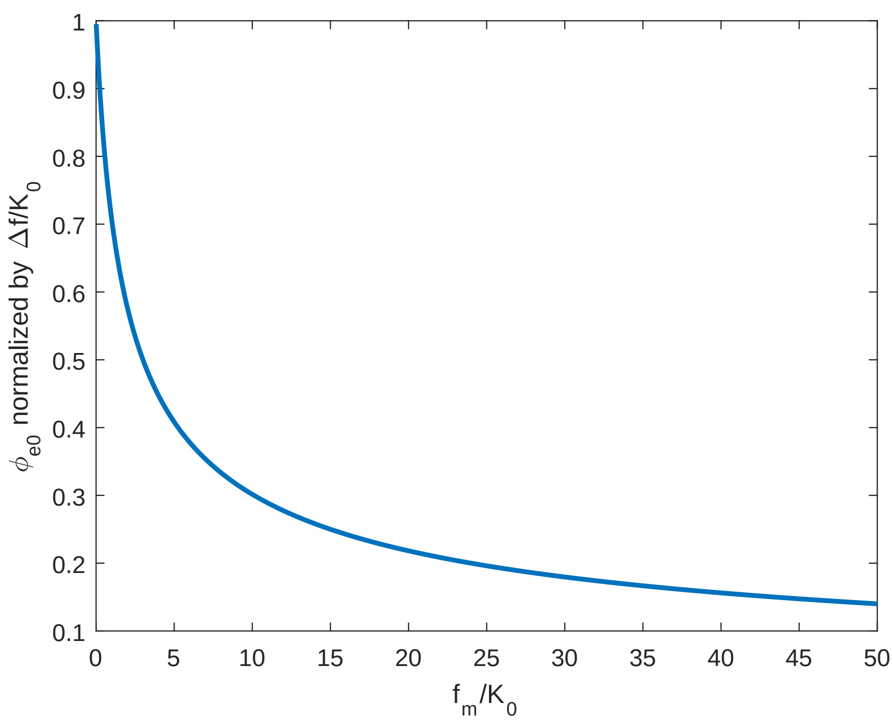

Figure 1: Normalized ϕ_{e0} w.r.t. $\Delta f/K_0$ versus f_m/K_0

d. Since we require $|\phi_e(t)| \leq 0.5$ for the approximation $\sin\phi_e(t) \approx \phi_e(t)$ and $\phi_e(t) = \phi_{e0}\cos(2\pi f_m t + \psi)$, this implies $\Delta f/K_0 \leq 0.5$ is sufficient for the approximation.

Problem 2. This is a continuation of Problem 1. The output signal $v(t)$ of the loop is related to the phase error $\phi_e(t)$ by

$$v(t) = \frac{K_0}{k_v}\phi_e(t).$$

a. Let us write
$$v(t) = A_0 \cos(2\pi f_m t + \psi).$$
Give the equations for A_0 and ψ.

b. At what value of f_m will the amplitude of the loop output $v(t)$ have fallen by 3 dB?

c. Show that the input frequency range over which the loop will hold lock is $\pm K_0$.

Solution:

a. From the previous problem, we have

$$\phi_e(t) = \phi_{e0}\cos(2\pi f_m t + \psi),$$

where $\phi_{e0} = \frac{\Delta f/K_0}{\sqrt{1+(f_m/K_0)^2}}$ and $\psi = \tan^{-1}\left(\frac{-f_m}{K_0}\right)$. Therefore, given $v(t) = \frac{K_0}{k_v}\phi_e(t)$, we conclude that

$$A_0 = \frac{\Delta f/k_v}{\sqrt{1+(f_m/K_0)^2}},$$
$$\psi = \tan^{-1}\left(\frac{-f_m}{K_0}\right).$$

b. Since $v(t) = A_0\cos(2\pi f_m t + \psi)$ and $A_0 = \frac{\Delta f/k_v}{\sqrt{1+(f_m/K_0)^2}}$, we see that the amplitude of the loop output $v(t)$ has a 3dB drop (amplitude divided by two) if $f_m/K_0 = 1$. Hence, we have a 3dB fall at $f_m = K_0$.

c. Given input signal with generalized angle $\phi_1(t)$, we have its instantaneous frequency
$$f_i(t) = \frac{1}{2\pi}\frac{d\phi_1(t)}{dt}.$$

Knowing that the phase remains locked if $\frac{d\phi_e(t)}{dt} = 0$ and the first-order PLL model as

$$\frac{d\phi_e(t)}{dt} + 2\pi K_0[\sin\phi_e(t) * \delta(t)] = \frac{d\phi_1(t)}{dt},$$

we have

$$0 + 2\pi K_0 \sin\phi_e(t) = 2\pi f_i(t).$$

Hence, the instantaneous frequency $f_i(t) = K_0 \sin\phi_e(t) \in [-K_0, K_0]$.

Problem 3. Consider an FM wave of carrier frequency f_c which is produced by a modulating wave $m(t)$. Assume that f_c is large enough to justify treating this FM wave as a narrow-band signal.

a. Find an approximate expression for its Hilbert transform.

b. For the special case of a sinusoidal wave $m(t) = A_m \cos(2\pi f_m t)$, find the exact expression for the Hilbert transform of the resulting FM wave. For this case, what is the error in the approximation used in part (a)?

Solution:

a. Let the FM signal be $s(t) = A_c \cos(2\pi f_c t + \phi(t))$, where $\phi(t) = 2\pi k_f \int_0^t m(s)ds$. We can thus find its Hilbert transform $\hat{s}(t)$ by considering its frequency response

$$\begin{aligned}
s(t) &= A_c \cos(2\pi f_c t)\cos\phi(t) - A_c \sin(2\pi f_c t)\sin\phi(t) \\
\overset{\mathcal{F}}{\leftrightarrow} S(f) &= \frac{A_c}{2}\left[\delta(f-f_c) + \delta(f+f_c)\right] * \Phi_1(f) - \frac{A_c}{2j}\left[\delta(f-f_c) - \delta(f+f_c)\right] * \Phi_2(f) \\
&= \frac{A_c}{2}\left[\Phi_1(f-f_c) + \Phi_1(f+f_c)\right] - \frac{A_c}{2j}\left[\Phi_2(f-f_c) - \Phi_2(f+f_c)\right].
\end{aligned}$$

where $\cos\phi(t) \overset{\mathcal{F}}{\leftrightarrow} \Phi_1(f)$, $\sin\phi(t) \overset{\mathcal{F}}{\leftrightarrow} \Phi_2(f)$ and they are both narrowband signals.

Then the Hilbert transformer makes positive spectrum $f \geq 0$ multiplied by $-j$ negative spectrum $f < 0$ multiplied by j. Hence,

$$\begin{aligned}
\hat{S}(f) &= \frac{A_c}{2}\left[-j\Phi_1(f-f_c) + j\Phi_1(f+f_c)\right] - \frac{A_c}{2j}\left[-j\Phi_2(f-f_c) - j\Phi_2(f+f_c)\right] \\
&= \frac{A_c}{2j}\left[\Phi_1(f-f_c) - \Phi_1(f+f_c)\right] + \frac{A_c}{2}\left[\Phi_2(f-f_c) + \Phi_2(f+f_c)\right] \\
\overset{\mathcal{F}}{\leftrightarrow} \hat{s}(t) &= A_c \sin(2\pi f_c t)\cos\phi(t) + A_c \cos(2\pi f_c t)\sin\phi(t).
\end{aligned}$$

b. Suppose that the modulating signal $m(t) = A_m \cos(2\pi f_m t)$, then $\phi(t) = \frac{k_f A_m}{f_m} \sin(2\pi f_m t) = \beta \sin(2\pi f_m t)$. We thus have the modulated signal as

$$s(t) = A_c \cos(2\pi f_c t) \cos(\beta \sin(2\pi f_m t)) - A_c \sin(2\pi f_c t) \sin(\beta \sin(2\pi f_m t)),$$

and its Hilbert transform as

$$\hat{s}(t) = A_c \sin(2\pi f_c t) \cos(\beta \sin(2\pi f_m t)) - A_c \cos(2\pi f_c t) \sin(\beta \sin(2\pi f_m t)).$$

Now we assume that $s(t)$ is a narrowband FM signal, which gives us an approximation as

$$\begin{aligned} s_a(t) &= A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_c t) \sin(2\pi f_m t) \\ \overset{\mathcal{F}}{\leftrightarrow} S_a(f) &= \frac{A_c}{2} [\delta(f - f_c) + \delta(f + f_c)] + \frac{\beta A_c}{4} [\delta(f - f_c - f_m) - \delta(f - f_c + f_m) \\ &\quad - \delta(f + f_c - f_m) + \delta(f + f_c + f_m)]. \end{aligned}$$

Then,

$$\begin{aligned} \hat{S}_a(f) &= \frac{A_c}{2} [-j\delta(f - f_c) + j\delta(f + f_c)] + \frac{\beta A_c}{4} [-j\delta(f - f_c - f_m) + j\delta(f - f_c + f_m) \\ &\quad - j\delta(f + f_c - f_m) + j\delta(f + f_c + f_m)] \\ &= \frac{A_c}{2j} [\delta(f - f_c) - \delta(f + f_c)] + \frac{\beta A_c}{4j} [\delta(f - f_c - f_m) - \delta(f - f_c + f_m) \\ &\quad + \delta(f + f_c - f_m) - \delta(f + f_c + f_m)] \\ \overset{\mathcal{F}}{\leftrightarrow} \hat{s}_a(t) &= A_c \sin(2\pi f_c t) + \beta A_c \cos(2\pi f_c t) \sin(2\pi f_m t) \end{aligned}$$

Finally, the error between the exact Hilbert transform and the approximated one is

$$e(t) = \hat{s}(t) - \hat{s}_a(t).$$

Problem 4. Let $X_k,\ k = 1, 2, \ldots$ be independent Bernoulli distributed random variables such that

$$P(X_k = 1) = p, \quad P(X_k = 0) = 1 - p, \quad k = 1, 2, \ldots, \quad 0 < p < 1.$$

Let

$$Y_m = \prod_{k=1}^{m} X_k, \quad m \geq 1.$$

a. Compute $E[Y_m]$ for any $m \geq 1$ and also find $\lim_{m\to\infty} E[Y_m]$.

b. Compute $Var(Y_m)$ for any $m \geq 1$ and also find $\lim_{m\to\infty} Var(Y_m)$.

c. Compute $E[Y_m Y_n]$ for any $m, n \geq 1$ and also deduce the limiting behavior of $E[Y_m Y_n]$ as m and n tend to infinity.

Solution:

a. Since the Bernoulli random variables X_k are independent $\forall k$,

$$E[Y_m] = E\left[\prod_{k=1}^{m} X_k\right] = \prod_{k=1}^{m} E\left[X_k\right] = p^m$$
$$\lim_{m\to\infty} E[Y_m] = 0, \text{ since } 0 < p < 1.$$

b. Since $\mathrm{Var}(Y_m) = E[Y_m^2] - (E[Y_m])^2$ and $E[Y_m^2] = \prod_{k=1}^{m} E[X_k^2] = p^m$, we have

$$\mathrm{Var}(Y_m) = p^m - p^{2m},$$
$$\lim_{m\to\infty} \mathrm{Var}(Y_m) = 0.$$

c. Assume without loss of generality $m \geq n$,

$$\begin{aligned} E[Y_m Y_n] &= E\left[\prod_{k=1}^{n} X_k^2 \prod_{l=n+1}^{m} X_l\right] \\ &= E\left[\prod_{k=1}^{n} X_k^2\right] E\left[\prod_{l=n+1}^{m} X_l\right] \\ &= \prod_{k=1}^{n} E[X_k^2] \prod_{l=n+1}^{m} E[X_l] \\ &= p^n \cdot p^{m-n} = p^n, \text{ if } m \geq n. \end{aligned}$$

Hence, $E[Y_m Y_n] = p^{\max(m,n)}$, and

$$\lim_{m\to\infty} \lim_{n\to\infty} E[Y_m Y_n] = 0.$$