EE 567

Homework 1 Solution

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Problem 1. Compute the Fourier transform of the pulse, $x(t)$, defined by

$$x(t) = \begin{cases} 1-t, & 0 < t \le 1, \\ 0, & elsewhere \end{cases}$$

and sketch a plot of its graph in the frequency domain. You should produce two plots (one for the magnitude of the Fourier transform and one for the phase). In the magnitude plot, clearly identify where the maximum height occurs and the value of the maximum height. Also, identify where the first zero-crossings closest to the peak magnitude occur.

Solution:
Using the definition of Fourier transform

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt,$$

we have

$$\begin{aligned} X(f) &= \int_0^1 (1-t) e^{-j2\pi ft} dt \\ &= \left[(1-t)\left(-\frac{1}{j2\pi f}\right) e^{j2\pi ft} \right]_{t=0}^{1} - \int_0^1 (-1)\left(-\frac{1}{j2\pi f}\right) e^{-j2\pi ft} dt \\ &= \frac{1}{j2\pi f} + \frac{1}{(j2\pi f)^2} e^{-j2\pi ft} \bigg|_{t=0}^{1} \\ &= \frac{1}{j2\pi f} + \frac{1-e^{-j2\pi f}}{(2\pi f)^2}, \end{aligned}$$

where the second line comes from integration by parts:

$$\int_a^b u(t)v'(t)dt = u(t)v(t)\bigg|_a^b - \int_a^b u'(t)v(t)dt.$$

Problem 2. Use direct integration to find the Fourier transform of the signal

$$g(t) = \exp(-|t|), \quad -\infty < t < \infty.$$

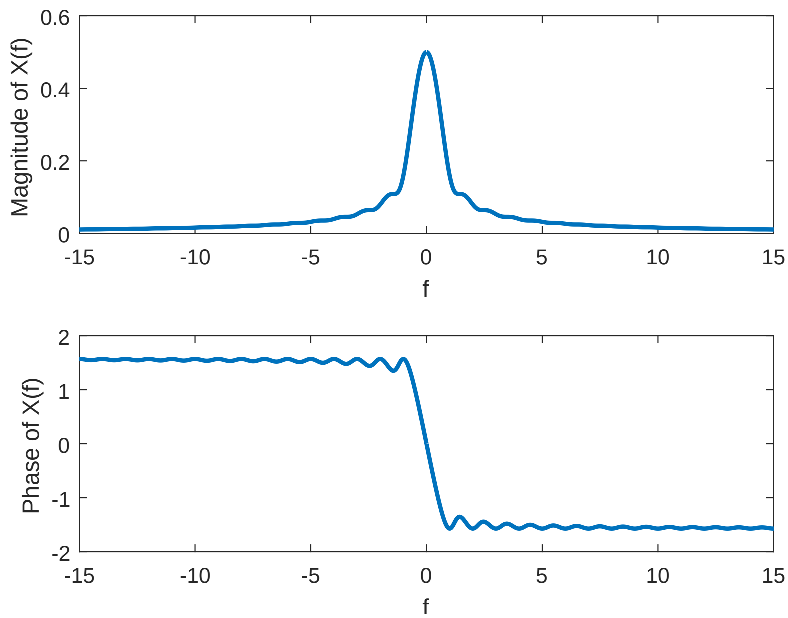

Figure 1: The Fourier transform of $x(t)$ attains its maximum at $f = 0$ with $|X(0)| = 0.5$ and its first zero-crossing is at $f = \infty$ (or no zero-crossing).

Solution:

$$
\begin{aligned}
G(f) &= \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{0} e^{t}e^{-j2\pi ft}dt + \int_{0}^{\infty} e^{-t}e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{0} e^{(1-j2\pi f)t}dt + \int_{0}^{\infty} e^{(-1-j2\pi f)t}dt \\
&= \frac{1}{1-j2\pi f}e^{(1-j2\pi f)t}\Big|_{t=-\infty}^{0} + \frac{1}{-1-j2\pi f}e^{(-1-j2\pi f)t}\Big|_{t=0}^{\infty} \\
&= \frac{1}{1-j2\pi f} + \frac{1}{1+j2\pi f} \\
&= \frac{2}{1+4\pi^2 f^2}
\end{aligned}
$$

Problem 3. Determine which of the following systems is linear.

a. $y(t) = x(t)$.

b. $y(t) = x^2(t)$.

c. $y(t) = c_1 x(t+1) + c_2 x(t-1), \quad c_1, c_2 \neq 0$.

d. $y(t) = t \cdot x(t),\ t > 0$.

Solution:
A system is linear if it possesses additivity property and homogeneity property, that is, suppose we have input/output pairs $x_1(t) \to y_1(t)$ and $x_2(t) \to y_2(t)$, the output of the weighted sum of inputs $x(t) = \alpha x_1(t) + \beta x_2(t)$ is

$$y(t) = \alpha y_1(t) + \beta y_2(t)$$

which is the weighted sum of individual outputs.

a. Let $x(t) = \alpha x_1(t) + \beta x_2(t)$, then the corresponding output

$$\begin{aligned} y(t) &= x(t) \\ &= \alpha x_1(t) + \beta x_2(t) \\ &= \alpha y_1(t) + \beta y_2(t). \end{aligned}$$

It is a linear system.

b. Let $x(t) = \alpha x_1(t) + \beta x_2(t)$, then

$$\begin{aligned} y(t) &= x^2(t) \\ &= (\alpha x_1(t) + \beta x_2(t))^2 \\ &\neq \alpha y_1(t) + \beta y_2(t). \end{aligned}$$

Hence, it is not a linear system.

c. Let $x(t) = \alpha x_1(t) + \beta x_2(t)$, then

$$\begin{aligned} y(t) &= c_1 x(t+1) + c_2 x(t-1) \\ &= c_1 \left[\alpha x_1(t+1) + \beta x_2(t+1)\right] + c_2 \left[\alpha x_1(t-1) + \beta x_2(t-1)\right] \\ &= \alpha[c_1 x_1(t+1) + c_2 x_1(t-1)] + \beta[c_1 x_2(t+1) - c_2 x_2(t-1)] \\ &= \alpha y_1(t) + \beta y_2(t). \end{aligned}$$

It is a linear system.

d. Let $x(t) = \alpha x_1(t) + \beta x_2(t)$, then $\forall t > 0$

$$\begin{aligned} y(t) &= t x(t) \\ &= t[\alpha x_1(t) + \beta x_2(t)] \\ &= \alpha y_1(t) + \beta y_2(t). \end{aligned}$$

It is a linear system.

Problem 4. Find the constant k such that

$$\int_{-\infty}^{\infty} \text{sinc}(kt) dt = 1.$$

Solution:
In this problem, we assume the sinc$(\cdot)$ is the normalized sinc function:

$$\text{sinc}(x) = \frac{\sin \pi x}{\pi x},$$

and the rectangular function is 1 within $[-1/2, 1/2]$,

$$\text{rect}(x) = \begin{cases} 1, & \text{if } x \in [-1/2, 1/2] \\ 0, & \text{otherwise.} \end{cases}$$

To find the value k so that the integral equals 1, we consider the Fourier transform pair of rectangular function and sinc function, i.e. $\text{sinc}(at) \leftrightarrow$

$\frac{1}{|a|}\text{rect}(\frac{f}{a})$. Then we write down the Fourier transform equation with $f = 0$ as follow

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \\
\Rightarrow \frac{1}{|k|}\text{rect}\left(\frac{f}{k}\right) &= \int_{-\infty}^{\infty} \text{sinc}(kt)e^{-j2\pi ft}dt \\
\Rightarrow \frac{1}{|k|} \cdot 1 &= \int_{-\infty}^{\infty} \text{sinc}(kt)dt
\end{aligned}
$$

Hence, we conclude that in order to satisfy the integral, we need $k = \pm 1$.