EE 567

Homework 10 Solutions

Problem 1 (from EE 567 Final, Fall 2014). A binary communication system is used where at the receiver the test statistic is $z(T) = ca_i + n_0,\ i = 1, 2$ where $a_1 = +1$ and $a_2 = -1$ and $c = 1$ if a signal is present and $c = 0$ if a signal is not present. The noise n_0 is distributed as

$$p(n_0) = \begin{cases} n_0 + 1.0, \ -1.0 \leq n_0 \leq 0 \\ \\ -(n_0 - 1.0), \ 0 \leq n_0 \leq 1.0 \\ \\ 0 \text{ elsewhere.} \end{cases}$$

However, the receiver is just interested in detecting whether or not the signal is actually present. If the signal is not present the noise n_0 is still present and if the signal is present the noise is present also. So, the receiver designer decides to take the absolute value of the test statistic and compare this to a threshold. If $|Z(T)| > T_0$, for some T_0, then a signal is declared present, otherwise, it is not.

a. Find T_0 such that the probability of false alarm is 10^{-4}.

   Solution: A graph of the density function is a triangle centered at zero. An equation of the right side of the absolute value of the density is $y = -2z + 2$. Thus, we solve

   $$\frac{1}{2}(1 - T_0)(-2T_0 + 2) = 10^{-4}$$

   to get $T_0 = 0.99$.

b. If a signal is present find the probability that you would detect it with this logic if the value of $T_0 = 3/4$ (which is not the answer to part (a)).

   Solution: With a signal present the pdf $f(z)$ is a triangle centered at ± 1 and after taking absolute values it is a triangle centered at 1 with region of support extending from 0 to 2. The equation of the left side of the triangle is $y = z$. Thus,

   $$P_d = \int_{3/4}^{2} f(z)dz = 1\int_{3/4}^{1} zdz + \frac{1}{2} = \frac{23}{32} = 0.72.$$

Problem 2 (from EE 567 Final, Fall 2014). Explain the concept of M of N detection. How does this compare (in terms of dB performance and implementation complexity) to the usual integration detection? Also, give the formula you should use to calculate the overall probability of detection given that the output of the energy detector before the M of N logic exceeds the detection threshold with probability $p_{d,s}$. For a given value of N what is a good value of M to use?

Solution: In the detection scheme we look for M of more threshold crossings after the energy detector. The performance is about 1 dB less than integration detection. The overall probability of detection is computed as

$$P_d = \sum_{k=M}^{N} \binom{N}{k} p_{d,s}^k (1 - p_{d,s})^{N-k}.$$

The optimal value for M is approximately N/2 where we round up if N is odd.

Problem 3 (revised from EE 567 Final, Fall 2014). Suppose we receive a BPSK signal of the form

$$r(t) = A_1 \cos(2\pi f_c t) + n(t), \quad 0 \leq t \leq T$$

where $n(t)$ is a noise process, $T = 1$ sec, $f_c = 1$ kHz and $A_1 = \pm 2$ with equal probability. The signal is demodulated in an optimal manner and then an optimal threshold is used in making a decision. For demodulation assume the received signal $r(t)$ is mixed with a cosine wave of amplitude 1 with the same frequency and phase of the transmitted signal of interest and then low pass filtered by integrating from 0 to T. With this approach suppose at the output of the demodulator (just prior to the threshold comparator) the noise (call it n) is characterized by the Laplacian density

$$p(n) = \frac{1}{\sqrt{2}\sigma} e^{-|n|\sqrt{2}/\sigma}, \quad (-\infty < n < \infty)$$

where we will take $\sigma = 1$.

a. Using the optimal detector (with the corresponding optimal threshold) as described above compute the probability of a correct decision, P_c, for the value of A_1. You may assume that f_c is known to the receiver.

Solution: Clearly, the optimal threshold is $T = 0$. We can assume WLOG that $A_1 = 2$. After mixing and applying the LPF the density of the signal plus noise is centered at $A_1/2 = 1$. Thus,

$$\begin{aligned} P_c &= \frac{1}{\sqrt{2}} \int_0^\infty e^{-|x-1|\sqrt{2}} dx \\ &= \frac{1}{\sqrt{2}} \int_0^1 e^{(x-1)\sqrt{2}} dx + \frac{1}{\sqrt{2}} \int_1^\infty e^{-(x-1)\sqrt{2}} dx \\ &= 1 - \frac{1}{2} e^{-\sqrt{2}} = 0.88. \end{aligned}$$

b. Now suppose an interfering signal is also present so that

$$r(t) = A_1 \cos(2\pi f_c t) + A_2 \cos(2\pi f_c t + \pi/4) + n(t) \quad 0 \le t \le T.$$

However, the receiver does not know the interfering signal is present so the receiver does not change the threshold used in part (a) nor does the receiver make any other modifications. Find the value of A_2 so that if a symbol is transmitted with $A_1 = 2$ then the probability of a correct decision for that symbol is now 0.1.

Solution: Now after mixing and low pass filtering the signal plus interferer plus noise is centered at

$$\begin{aligned} \mu &= \frac{A_1}{2} + \frac{A_2}{2} \cos(\pi/4) \\ \mu &= 1 + \frac{A_2}{2} \cos(\pi/4). \end{aligned}$$

We thus solve

$$\begin{aligned} P_c &= \frac{1}{\sqrt{2}} \int_0^\infty e^{-|x-\mu|\sqrt{2}} dx = 0.1 \\ &= \frac{1}{\sqrt{2}} \int_0^\mu e^{(x-\mu)\sqrt{2}} dx + \frac{1}{\sqrt{2}} \int_\mu^\infty e^{-(x-\mu)\sqrt{2}} dx = 0.1 \\ &= 1 - \frac{1}{2} e^{-\mu\sqrt{2}} = 0.1 \end{aligned}$$

which gives

$$\mu = -\frac{2\ln 0.9}{\sqrt{2}} = 0.149$$

and then

$$1 + \frac{A_2}{2}\cos(\pi/4) = 0.149$$

or

$$A_2 = -2.41.$$

c. Using the value of A_2 found in part (b) find the probability of a correct decision at the receiver assuming $A_1 = -2$.

Solution: Now after mixing and low pass filtering the signal plus interferer plus noise is centered at

$$\begin{aligned}\mu &= -1 - \frac{2.41}{2}\cos(\pi/4) \\ &= -1.851.\end{aligned}$$

We compute

$$\begin{aligned}P_c &= \frac{1}{\sqrt{2}}\int_{-\infty}^{0} e^{-|x-\mu|\sqrt{2}}dx \\ &= \frac{1}{\sqrt{2}}\int_{-\infty}^{\mu} e^{(x-\mu)\sqrt{2}}dx + \frac{1}{\sqrt{2}}\int_{\mu}^{0} e^{-(x-\mu)\sqrt{2}}dx \\ &= 1 - \frac{1}{2}e^{\mu\sqrt{2}} = 0.964.\end{aligned}$$

d. Combine parts (b) and (c) to compute the overall probability of a correct decision in the presence of the interferer.

Solution: With equally likely signals we find

$$P_c = (0.1 + 0.964)\frac{1}{2} = 0.532.$$

e. Again using the value of A_2 found in part (b) find the overall probability of a correct decision if the interferer's frequency is changed such that now $f_0 = 0.8$ kHz and thus

$$r(t) = A_1 \cos(2\pi f_c t) + A_2 \cos(2\pi f_0 t + \pi/4) + n(t) \quad 0 \le t \le T.$$

Solution: Now after mixing and low pass filtering the signal plus interferer plus noise is centered at

$$\begin{aligned}
\mu &= \int_0^T \left(\frac{A_1}{2} + \frac{A_2}{2}\cos(2\pi(f_c - f_0)t + \pi/4)\right) dt \\
&= \frac{A_1}{2} + \int_0^1 \left(\frac{A_2}{2}\cos(2\pi(f_c - f_0)t + \pi/4)\right) dt \\
&= \frac{A_1}{2} + \int_0^1 \left(\frac{A_2}{2}\cos(2\pi 200 t + \pi/4)\right) dt \\
&= \frac{A_1}{2}.
\end{aligned}$$

Thus, $P_c = 0.88$ as found in part (a).

Problem 4 **(from Sklar, Digital Communications, 2nd ed.).** Suppose an airplane terminal communicating with a satellite is equipped with a frequency hopping spread spectrum system transmitting with an EIRP of 20 dBW (dB Watts). The data rate is $R = 100$ bits/sec. The jammer is transmitting wideband Gaussian noise continually with $\text{EIRP}_J = 60$ dBW. Assume that $(E_b/J_0)_{reqd} = 10$ dB and that the path loss is identical for both the airplane terminal and the jammer.

a. Should the communicators be more concerned with the jammer trying to jam the uplink or the downlink?

   Solution: The jamming of the uplink is of more concern since the jammer could degrade the communications of multiple terminals that are using the satellite. To achieve the same degradation on the downlink the jammer would have to jam each of the receiving terminals.

b. If it is desired to have an AJ margin of 20 dB, what should be the value of the hopping bandwidth W_{ss}?

   Solution: In this case we have

$$M_{AJ}\ (\text{dB}) = \left(\frac{J}{S}\right)_{reqd} (\text{dB}) - \left(\frac{J}{S}\right)_r (\text{dB}).$$

   We will assume the path loss is the same for both the communicator and the jammer. We can then replace $\left(\frac{J}{S}\right)_r$ with the ratio of transmitted jammer-to-signal power. Hence,

$$\begin{aligned} M_{AJ}\ (\text{dB}) &= \left(\frac{J}{S}\right)_{reqd} (\text{dB}) - \text{EIRP}_J\ (\text{dBW}) + \text{EIRP}_T\ (\text{dBW}) \\ &= G_P\ (\text{dB}) - \left(\frac{E_b}{J_0}\right)_{reqd} (\text{dB}) - \text{EIRP}_J\ (\text{dBW}) \\ &\quad + \text{EIRP}_T\ (\text{dBW}). \end{aligned}$$

   Therefore,

$$G_P = 20\ (\text{dB}) + 10\ (\text{dB}) + 60\ (\text{dBW}) - 20\ (\text{dBW}) = 70\ (\text{dB})$$

   and

$$\begin{aligned} W_{ss} &= G_P\ (\text{dB}) + R\text{dB-Hz} = 70\ (\text{dB}) + 20\ (\text{dB-Hz}) \\ &= 90\ (\text{dB-Hz}) = 1\ (\text{GHz}). \end{aligned}$$

Problem 5 (from Sklar, Digital Communications, 2nd ed.). Assume that a repeat-back jammer is located $d = 30$ km away from the communicator. Assume further that the jammer can monitor any uplink transmission from the communicator to a nearby satellite. How fast must the communicator hop his frequency to evade the repeat-back jammer? Assume that the jammer can change its jamming frequnecy in zero time and that the only differential delay between the communicator's uplink signal and the jamming uplink signal is the propogation delay from the communicator to the jammer.

Solution: To evade the repeat-back jammer the communicator must ensure the transmission time at a particular frequency and the jammer's attempt to disrupt that transmission using that frequency do not overlap in time. Thus, the duration of each hop (how long the communicator dwells at a particular frequency) must satisfy

$$T_{hop} \leq \frac{d}{c} = \frac{3 \times 10^4 \text{ m}}{3 \times 10^8 \text{ m/sec}} = 10^{-4} \text{ sec}$$

where c is the speed of light. Thus, $R_{hop} \geq 10,000$ hops/sec.