Problem 4. Receiver Analysis.
A receiver front end has a noise figure of 10 dB and a gain of 50 dB and a bandwidth of 11 MHz. The input signal power is 10^{-11} W. The antenna temperature is 175 K. Find $T_e,\ T_s,\ N_{out},\ SNR_{in}$ and SNR_{out}. You may use $T_0 = 290$ K and Boltzmann's constant equals 1.38×10^{-23} J/K.

Solution: We find with F =noise figure, T_a = antenna temperature, G = gain, B_n = bandwidth, S_{in} = input power

$$
\begin{aligned}
T_e &= (10^{F/10} - 1)T_0 = 2610\ K \\
T_s &= T_a + T_e = 2785\ K \\
N_{in} &= kT_aB_n = 2.66 \times 10^{-14} \\
N_{out} &= kT_sB_n \times 10^{G/10} = 4.23 \times 10^{-8} \\
SNR_{in} &= \frac{S_{in}}{N_{in}} = 376 = 25.8\ dB \\
S_{out} &= S_{in} \times 10^{G/10} = 1.00 \times 10^{-6} \\
SNR_{out} &= \frac{S_{out}}{N_{out}} = 23.7 = 13.7\ dB.
\end{aligned}
$$

Problem 5. BPSK Signaling.
Suppose we receive a BPSK signal of the form

$$r(t) = A\cos(2\pi f_c t + \theta) + n(t), \quad 0 \le t \le T$$

where $n(t)$ is an AWGN process and $A = \pm 1$ equally likely. The signal is demodulated in an optimal manner and then an optimal threshold is used in making a decision. For demodulation assume the received signal $r(t)$ is mixed with a cosine wave with the same frequency and phase of the transmitted signal of interest and then low pass filtered by integrating from 0 to T. This is coherent detection with a matched filter. With this approach we derived in class the probability of bit error as

$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right).$$

a. Suppose in the mixing signal the receiver does not use the exact correct phase but instead uses $\cos(2\pi f_c t + \hat{\theta})$. Define $\Delta\theta = \theta - \hat{\theta}$. What is the correct expression for P_b in this case?

Solution: We derived that for antipodal BPSK signaling with equally likely symbols

$$P_b = \int_{(a_1+a_2)/2}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-(z-a_2)^2/2\sigma_0^2} dz$$

where a_1 is the mean of the test statistic $z(T)$ resulting after coherent detection when $A = +1$, i.e., $a_1 = |A|T/2$, and a_2 is the mean when $A = -1$, , i.e., $a_2 = -|A|T/2$, σ_0^2 is the variance of the noise in the test statistic. Substituting $u = \frac{z-a_2}{\sigma_0}$ we got

$$P_b = \int_{(a_1-a_2)/2\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

or

$$P_b = Q\left(\frac{a_1 - a_2}{\sigma_0}\right)$$

where

$$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du.$$

Letting $a_1 = \sqrt{E}$, $a_2 = -\sqrt{E}$ and $\sigma_0^2 = \frac{N_0}{2}$ we obtained the expression

$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

given above. We now let $a_1 = \sqrt{E}\cos\Delta\theta$, $a_2 = -\sqrt{E}\cos\Delta\theta$ to get

$$P_b = Q\left(\sqrt{\frac{2E_b}{N_0}}\cos\Delta\theta\right).$$

b. Now suppose $\Delta\theta = 0$ but we have an interferer or jammer present of the form

$$J(t) = A_J \cos(2\pi f_c t + \theta)$$

where, $A_J = 1/2$ always (does not change sign). So the receiver processes $r(t)+J(t)$. But the receiver does not know the jammer is present. Derive the expression for P_b in this case?

Solution: Assuming first that symbol 2 was sent we get

$$\begin{aligned} P_{b,2} &= \int_{(a_1+a_2)/2}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-(z-\tilde{a}_2)^2/2\sigma_0^2} dz \\ &= \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-(z-\tilde{a}_2)^2/2\sigma_0^2} dz \end{aligned}$$

where $\tilde{a}_2 = -\sqrt{E}/2$. Substituting $u = \frac{z-\tilde{a}_2}{\sigma_0}$ we get

$$P_{b,2} = \int_{-\tilde{a}_2/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} dz$$

or

$$P_{b,2} = Q\left(\sqrt{\frac{E_b}{2N_0}}\right).$$

If symbol 1 was sent we get

$$\begin{aligned} P_{b,1} &= \int_{-\infty}^{(a_1+a_2)/2} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-(z-\tilde{a}_1)^2/2\sigma_0^2} dz \\ &= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-(z-\tilde{a}_1)^2/2\sigma_0^2} dz \end{aligned}$$

where $\tilde{a}_1 = 3\sqrt{E}/2$. Substituting $u = \frac{z-\tilde{a}_1}{\sigma_0}$ we get

$$\begin{aligned} P_{b,1} &= \int_{-\infty}^{-\tilde{a}_1/\sigma_0} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} dz \\ &= \int_{\tilde{a}_1/\sigma_0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} dz \end{aligned}$$

or

$$P_{b,1} = Q\left(\sqrt{\frac{9E_b}{2N_0}}\right).$$

Since symbol 1 and symbol 2 are equally likely we get

$$P_b = P_{b,1} \times \frac{1}{2} + P_{b,2} \times \frac{1}{2}$$

or

$$P_b = Q\left(\sqrt{\frac{9E_b}{2N_0}}\right) \times \frac{1}{2} + Q\left(\sqrt{\frac{E_b}{2N_0}}\right) \times \frac{1}{2}.$$

c. Now say we have the ideal case again ($\Delta\theta = 0$ and no jammer) but we decide to implement a single bit error correction code, specifically, a (7,4) Hamming code. With P_b denoting the probability of an undecoded bit error, write down the mathematical expression you could use to compute the probability of an incorrect codeword after decoding (you do not have to evaluate this expression).

Solution: With P_{cw} denoting the probability of a codeword error then a codeword error occurs if we have 2 or more bits errors in a codeword so

$$P_{cw} = \sum_{k=2}^{7} \binom{n}{k} P_b^k (1 - P_b)^{7-k}.$$

Problem 6. A binary communication system is used where at the receiver the test statistic is $z(T) = ca_i + n_0,\ i = 1, 2$ where $a_1 = +1$ and $a_2 = -1$ (a_1 and a_2 are equally likely) and $c = 1$ if a signal is present and $c = 0$ if a signal is not present. The noise n_0 is distributed as

$$p(n_0) = \frac{1}{2\beta} e^{-|n_0|/\beta},\ -\infty < n_0 < \infty$$

where $\beta = 0.1$. However, the receiver is just interested in detecting whether or not the signal is actually present. If the signal is not present the noise n_0 is still present and if the signal is present the noise is present also. So, the receiver designer decides to take the absolute value of the test statistic and compare this to a threshold. If $|Z(T)| > T_0$, for some T_0, then a signal is declared present, otherwise, it is not.

a. Find T_0 such that the probability of false alarm is 10^{-3}.

   Solution: We find the density of $y = |Z|$ with noise only present is

$$p_Y(y) = \begin{cases} \frac{1}{\beta} e^{-y/\beta},\ 0 \le y < \infty, \\ 0, \quad \text{elsewhere.} \end{cases}$$

   We compute with only present

$$P_{fa} = \int_{T_0}^{\infty} \frac{1}{\beta} e^{-z/\beta} dz \Rightarrow T_0 = -\beta \ln(P_{fa}) = 0.691.$$

b. If a signal is present find the probability that you would detect it with this logic if the value of $T_0 = 0.75$ (which is not the answer to part (a)).

   Solution: We find the density of Z with signal plus noise present is

$$p_Z(z) = \frac{1}{2\beta} e^{-|z-1|/\beta},\ -\infty \le z < \infty.$$

   We find the density of $x = |Z|$ with signal plus noise present is then

$$p_X(x) = p_Z(x) + p_Z(-x), \quad x \ge 0$$

which becomes

$$p_X(x) = \begin{cases} \frac{1}{2\beta}e^{-|x-1|/\beta} + \frac{1}{2\beta}e^{-|-x-1|/\beta}, \ 0 \le x < \infty, \\ 0, \quad \text{elsewhere.} \end{cases}$$

We now compute

$$P_d = \int_{T_0}^{\infty} p_X(x)dx = 0.96.$$

A quicker solution is found by assuming without loss of generality that $a_1 = +1$ was sent. Then

$$P_d = \frac{1}{2} + \int_{T_0}^{1} \frac{1}{2\beta}e^{-|x-1|/\beta}dx = 0.96.$$