Problem 1. Phase Modulation.
Over the interval $0 \leq t \leq 1$ sec a PM signal is given by

$$s_{PM}(t) = 10\cos(2\pi f_0 t)$$

where $f_0 = 100$ kHz. It is known that the carrier frequency is 70 kHz. If $k_p = 1000$ rad/volt determine $m(t)$ over the interval $0 \leq t \leq 1$ sec.

Solution: An PM signal has phase $\theta(t)$ linear to the modulating signal $m(t)$, i.e.

$$\theta(t) = 2\pi f_c t + k_p m(t).$$

Hence, we have $s_{PM}(t) = A_c \cos(2\pi f_c t + k_p m(t)) = 10\cos 2\pi f_0 t$, which then gives us

$$\begin{aligned} k_p m(t) &= 2\pi(f_0 - f_c)t \\ \Rightarrow \quad & m(t) = \frac{2\pi \cdot 30 \cdot 10^3 t}{1000} = 60\pi t, t \in [0, 1]. \end{aligned}$$

Problem 2. Correlation Function.
Let $X_n,\ n = 1, 2, \ldots$ be independent normally (Gaussian) distributed random variables each with mean 0 and variance 1 (standard normal). Let

$$Y_m = \sum_{n=1+m}^{N+m} X_n^2, \quad m \geq 0.$$

Compute (analytically) the correlation $E[Y_0 Y_m]$ for $m = 0, 1, 2, \ldots$. You may use the fact that $E[X_n^4] = 3$.

Solution:

$$\begin{aligned} E[Y_0 Y_m] &= E\left[\left(\sum_{n=1}^{N} X_n^2\right)\left(\sum_{n=1+m}^{N+m} X_n^2\right)\right] \\ &= E\left[\sum_{j=1+m}^{N+m}\sum_{i=1}^{N} X_i^2 X_j^2\right] \\ &= \sum_{j=1+m}^{N+m}\sum_{i=1}^{N} E\left[X_i^2 X_j^2\right] \end{aligned}$$

which becomes

$$E[Y_0 Y_m] = \begin{cases} \sum_{j=1+m}^{N+m} \sum_{i=1, i\neq j}^{N} E[X_i^2]E[X_j^2] + \sum_{i=1+m}^{N} E[X_i^4], & m \leq N-1 \\ \sum_{i=1+m}^{N+m} \sum_{i=1}^{N} E[X_i^2]E[X_j^2], & m > N-1. \end{cases}$$

which simplifies to

$$E[Y_0 Y_m] = \begin{cases} N^2 + 2N - 2m, & m \leq N-1 \\ N^2, & m > N-1. \end{cases}$$

Problem 3. A/D Analysis.
A certain A/D converter has n physical bits. Suppose the input to the A/D is the waveform

$$s(t) = A\cos(2\pi f_c t) + n(t)$$

where $A\cos(2\pi f_c t)$ is the signal component and $n(t)$ is AWGN which at any time t has mean zero and variance σ^2. You can assume A and σ have units of volts.

a. Let us denote the maximum level of the A/D by M volts. We can apply a gain, G, to the input waveform to control clipping by the A/D. Find the maximum value of G so that if we sample the waveform when the signal component is at its peak value the probability of clipping is $P(clip) \leq 0.05$. You may use the fact that if Z is a standard normal random variable (mean zero, unit variance Gaussian), then $P(Z \geq z_\alpha) = \alpha$ where $z_\alpha = 1.645$ for $\alpha = 0.05$. Write your answer for G as a function of M, A and σ.

Solution: At some time $t = t_0$, the signal has the maximum value A and at that point the A/D sees $A + n(t) \sim N(A, \sigma^2)$. Applying the gain G we thus have $X = GA + Gn(t) \sim N(GA, G^2\sigma^2)$. We calculate

$$\begin{aligned} P(X \geq M) &= P\left(\frac{X - GA}{G\sigma} \geq \frac{M - GA}{G\sigma}\right) \\ &= P\left(Z \geq \frac{M - GA}{G\sigma}\right) = \alpha \end{aligned}$$

so

$$\frac{M - GA}{G\sigma} = z_\alpha$$

which yields

$$G = \frac{M}{A + z_\alpha \sigma}$$

where $z_\alpha = 1.645$.

b. Now suppose that instead of applying a gain G to control clipping we adjust the meaning of level changes within the A/D (so we set $G = 1$). Let use denote a level change by Δ volts (this is the quantization step size). For an n bit A/D converter find the largest value of Δ so that if we sample the waveform when the signal component is at its peak value the probability of clipping is $P(clip) \leq 0.05$. You should assume that an input value of zero volts lies directly in between the two middle levels of the A/D. Write your answer for Δ as a function of n, A and σ.

Solution: We sustitute

$$M = (2^n - 1)\frac{\Delta}{2}$$

in the expression for G found in part (a) and then setting $G = 1$ we find

$$\Delta = \frac{A + z_\alpha \sigma}{2^{n-1} - \frac{1}{2}}$$

where $z_\alpha = 1.645$.