EE 484

Spring 2016 Midterm Exam Solutions

Problem 1. [12 points]. Suppose $X_1, X_2, \ldots$ are each independent and normally distributed with mean zero and variance one (standard normal). Define $Y_1 = X_1$ and for $n = 2, 3, 4, \ldots$ let

$$Y_n = \alpha X_n + (1-\alpha)Y_{n-1}, \quad 0 < \alpha < 1.$$

a. Compute $E[Y_n Y_m]$ as a closed form function of n, m and α for $n \geq m$. Simplify as much as possible.

b. Evaluate your expression for $E[Y_n Y_m]$ using $n = 10$, $m = 5$ and $\alpha = 0.2$.

Solution:

$$\begin{aligned}
Y_1 &= X_1 \\
Y_2 &= \alpha X_2 + (1-\alpha)X_1 \\
&\vdots \\
Y_n &= \alpha X_n + (1-\alpha)\alpha X_{n-1} + (1-\alpha)^2 \alpha X_{n-2} + \cdots + (1-\alpha)^{n-2}\alpha X_2 + (1-\alpha)^{n-1} X_1
\end{aligned}$$

which may be written as

$$Y_n = \alpha \sum_{k=0}^{n-2} (1-\alpha)^k X_{n-k} + (1-\alpha)^{n-1} X_1.$$

Similarly,

$$Y_m = \alpha \sum_{j=0}^{m-2} (1-\alpha)^k X_{m-j} + (1-\alpha)^{m-1} X_1.$$

Since $E[X_n X_m] = 0$ for $n \neq m$ and $E[X_n^2] = 1$ we find for $n \geq m$

$$\begin{aligned}
E[Y_n Y_m] &= \alpha^2 \sum_{j=0}^{m-1} (1-\alpha)^{n-m+j} (1-\alpha)^j + (1-\alpha)^{n+m-2} \\
&= \alpha^2 (1-\alpha)^{n-m} \sum_{j=0}^{m-1} (1-\alpha)^{2j} + (1-\alpha)^{n+m-2}.
\end{aligned}$$

Computing the sum we find for $n \geq m$

$$E[Y_n Y_m] = \alpha^2 (1-\alpha)^{n-m} \frac{1-(1-\alpha)^{2m}}{1-(1-\alpha)^2} + (1-\alpha)^{n+m-2}.$$

Evaluating this expression for $n = 10,\ m = 5$ and $\alpha = 0.2$ we get

$$E[Y_{10} Y_5] = 0.08526.$$

Problem 2. [10 points]. Consider a real Gaussian random sequence $x(n)$, n an integer, with

$$E[x(n)] = 0, \quad E\left[x(n)^2\right] = 1, \quad E[x(n)x(m)] = \rho^{|n-m|}$$

where $0 < \rho < 1$. Let

$$y(n) = 2x(n) + 2.$$

a. Is $x(n)$ wide sense stationary?

   Solution: Yes.

b. Find the covariance of $y(n)$ and state whether or not it is wide sense stationary.

   Solution: $K_Y(n, m) = 4\rho^{|n-m|}$. Yes it is WSS.

Problem 3. [10 points]. Consider a wide-sense stationary random process $X(t)$ with autocorrelation function

$$R_X(\tau) = \begin{cases} 1 - \dfrac{|\tau|}{T}, & |\tau| \leq T \\ 0, & |\tau| > T. \end{cases}$$

Find $S_X(f)$, the power spectral density of $X(t)$.

Solution: We recognize that $R_X(\tau)$ is the convolution of a rectangular function with itself. Let

$$Rect(\tau) = \begin{cases} 1, & |\tau| \leq T/2 \\ 0, & |\tau| > T/2. \end{cases}$$

Then

$$\begin{aligned} S_X(f) = F[R_X(\tau)] &= F\left[\frac{1}{\sqrt{T}} Rect(\tau) * \frac{1}{\sqrt{T}} Rect(\tau)\right] \\ &= F\left[\frac{1}{\sqrt{T}} Rect(\tau)\right] \cdot F\left[\frac{1}{\sqrt{T}} Rect(\tau)\right]. \end{aligned}$$

Now

$$F\left[\frac{1}{\sqrt{T}} Rect(\tau)\right] = \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} e^{-i2\pi f t} dt = \frac{1}{\sqrt{T}} \frac{\sin \pi f T}{\pi f}.$$

Hence,

$$S_X(f) = \frac{1}{T} \left(\frac{\sin \pi f T}{\pi f}\right)^2.$$

Problem 4. [6 points]. What are the two primary sources of error that lead to pointing losses by an antenna?

Solution: The primary sources of pointing error are imprecise knowledge of where the receiver is that you want to point to and also the inability of pointing exactly where you are trying to point due to imperfect control of mechanisms used to point the antenna.

Problem 5. [6 points]. Suppose a certain system that utilizes an A/D converter has 7 effective bits. It has been determined that 8 effective bits are needed. One could use another A/D coverter that has more physical bits. However, if that is not possible one can require that SINAD increase. How much would SINAD have to increase to obtain 8 effective bits instead of the current 7 effective bits?

Solution:

$$\text{Effective number of bits} = \frac{SINAD - 1.76}{6.02}$$

so we need to increase $SINAD$ by 6.02 dB to get another effective bit.

Problem 6. [16 points]. Consider a signal detector with an input

$$r = \pm A + n$$

where $+A$ and $-A$ occur with equal probability and the noise variance n is characterized by the Laplacian density

$$p(n) = \frac{1}{\sqrt{2}\sigma} e^{-|n|\sqrt{2}/\sigma}.$$

a. Derive the probability of error as a function of A and σ.

   Solution: $P_e = \frac{1}{2} e^{-\sqrt{2}A/\sigma}$.

b. Find the SNR ($= A^2/\sigma^2$) required to achieve an error probability of 10^{-5}.

   Solution: $P_e = \frac{1}{2} e^{-\sqrt{2\text{SNR}}} = 10^{-5} \Rightarrow \text{SNR} = 17.67\text{dB}$.

c. Repeat part (a) for the Gaussian density

$$p(n) = \frac{1}{\sqrt{2\pi}\sigma} e^{-n^2/2\sigma^2}.$$

Solution: $P_e = Q\left(\sqrt{\text{SNR}}\right)$.

d. Repeat part (b) for the above Gaussian density and compare the results you got using the Laplacian and Gaussian densities.

Solution: $P_e = 10^{-5} \Rightarrow \text{SNR} = 12.59\text{dB}$ so is 5 dB less than the Laplacian case

Problem 7. [10 points]. A receiver front end has a noise figure of 8 dB and a gain of 60 dB and a bandwidth of 7 MHz. The input signal power is 10^{-11} W. The antenna temperature is 165 K.

a. Find $T_e,\ T_s,\ N_{out},\ SNR_{in}$ and SNR_{out}.

Solution:

$$\begin{aligned}
T_e &= (F-1)\cdot 290\text{ K} = \left(10^{8/10}-1\right)\cdot 290\text{ K} = 1540\text{ K} \\
T_s &= T_a + T_e = 165\text{ K} + 1540\text{ K} = 1705\text{ K} \\
N_{out} &= G\cdot k\cdot B\cdot T_s = 1.65\times 10^{-7} \\
SNR_{in} &= \frac{S_i}{kT_aB} = 627.39 = 27.98\text{ dB} \\
SNR_{out} &= \frac{S_i\cdot G}{N_{out}} = 60.72 = 17.83\text{ dB.}
\end{aligned}$$

b. Repeat part [a] but using an antenna temperature of 8000 K (the sun is in the field of view).

Solution:

$$\begin{aligned}
T_e &= 1540\text{ K} \\
T_s &= T_a + T_e = 8000\text{ K} + 1540\text{ K} = 9540\text{ K} \\
N_{out} &= 9.22\times 10^{-7} \\
SNR_{in} &= 12.94 = 11.12\text{ dB} \\
SNR_{out} &= 10.85 = 10.35\text{ dB.}
\end{aligned}$$

Problem 8. [12 points]. Suppose we have BPSK modulation and we implement an error correction code, say, a (15,7) binary BCH code ($n = 15,\ k = 7$). This code can correct 2 bit errors in a block of size 15 bits. Plot on the same graph P_b vs. E_b/N_0 for both the uncoded and coded waveform. Your E_b/N_0 values should be in dB on the plot and should cover a range starting at 0 dB and extending far enough so that $P_b = 10^{-12}$ appears on the graph. Your plot should be done using a software package such as Excel or Matlab. Your P_b values should be plotted using a log scale.

Solution: For the uncoded case we have

$$P_{b,uc} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right).$$

For the coded case we have

$$P_{b,c} = \frac{1}{15}\sum_{k=3}^{15} k\binom{15}{k} P_{b,uc}^{k}(1 - P_{b,uc})^{15-k}.$$

In plotting the coded case we have to account for the overhead by adjusting the uncoded E_b/N_0 (dB) scale by $10\log_{10}\left(\frac{15}{7}\right) = 3.31$ (dB) or else account for the E_b/N_0 change directly in the evaluation of the Q-function. See the plot below.

Problem 9. [18 points]. Suppose we have an 8-bit A/D converter. Let us number these 256 levels from 1 to 256. Then the 0 voltage level lies halfway between levels 128 and 129. Suppose we have an input to the A/D consisting of two sinusoidal signals and noise of the form

$$s(t) = A_1\cos(2\pi f_1 t) + A_2\cos(2\pi f_2 t) + n(t)$$

where $n(t)$ is AWGN. Suppose that $A_1 = 10$, $A_2 = 5$, $f_1 = 2000$ Hz and $f_2 = 3000$ Hz. Assume that at any particular sample time $n(t)$ is mean 0 with variance σ^2 and is independent from sample to sample. Suppose each level of the A/D corresponds to a 1 volt change. Say we have a gain stage at the input to the A/D that we can apply to $s(t)$ to form $cs(t)$. We would like to find the maximum value of the constant c so that the output of the A/D from time 0 to infinity will clip no more than one percent of the time.

Note that this problem is *not* asking that the probability of clipping at the peak of $s(t)$ is 0.01. Rather, it is asking for the maximum value of c so that the output observed over a long time will clip no more than one percent of the time.

a. Using a sample rate of f_s Hz, give an analytical expression that one would have to evaluate to determine if a particular c would work, in general.

b. Suppose the composite signal $s(t)$ is sampled at 15000 Hz and the noise variance is $\sigma^2 = 1$. Find the numerical value of c required in the problem.

c. Suppose the composite signal $s(t)$ is sampled at 150000 Hz and the noise variance is $\sigma^2 = 1$. Find the numerical value of c required in the problem.

Note: If you use Matlab to aid you in part [b] and/or part [c] remember to hand in your code. Your Matlab code should only be used to evaluate analytic expressions (possibly iteratively until the correct c is found), not to perform a simulation where you simulate $s(t)$. Credit will not be given for answers obtained via a simulation of $s(t)$.

Solution: Letting $s(n) = s(nT_s) = s(n/f_s)$ we obtain

$$s(n) = A_1 \cos(2\pi n f_1/f_s) + A_2 \cos(2\pi n f_2/f_s) + n(n/f_s).$$

At sample n, $X(n) = cs(n)$ is normally distributed with mean $\mu_X(n) = cA_1 \cos(2\pi n f_1/f_s) + cA_2 \cos(2\pi n f_2/f_s)$ and variance $\sigma_X^2 = c^2\sigma^2$. The maximum A/D level before clipping is Max=127.5 and the minimum level is Min=-127.5. Thus

$$\begin{aligned} P(X(n) > \text{Max}) &= P\left(\frac{X(n) - \mu_X(n)}{\sigma_X} > \frac{\text{Max} - \mu_{\text{X}}(\text{n})}{\sigma_X}\right) \\ &= Q\left(\frac{\text{Max} - \mu_{\text{X}}(\text{n})}{\sigma_X}\right) \end{aligned}$$

and

$$
\begin{aligned}
P(X(n) < \text{Min}) &= P\left(\frac{X(n) - \mu_X(n)}{\sigma_X} < \frac{\text{Min} - \mu_{\text{X}}(\text{n})}{\sigma_X}\right) \\
&= 1 - Q\left(\frac{\text{Min} - \mu_{\text{X}}(\text{n})}{\sigma_X}\right).
\end{aligned}
$$

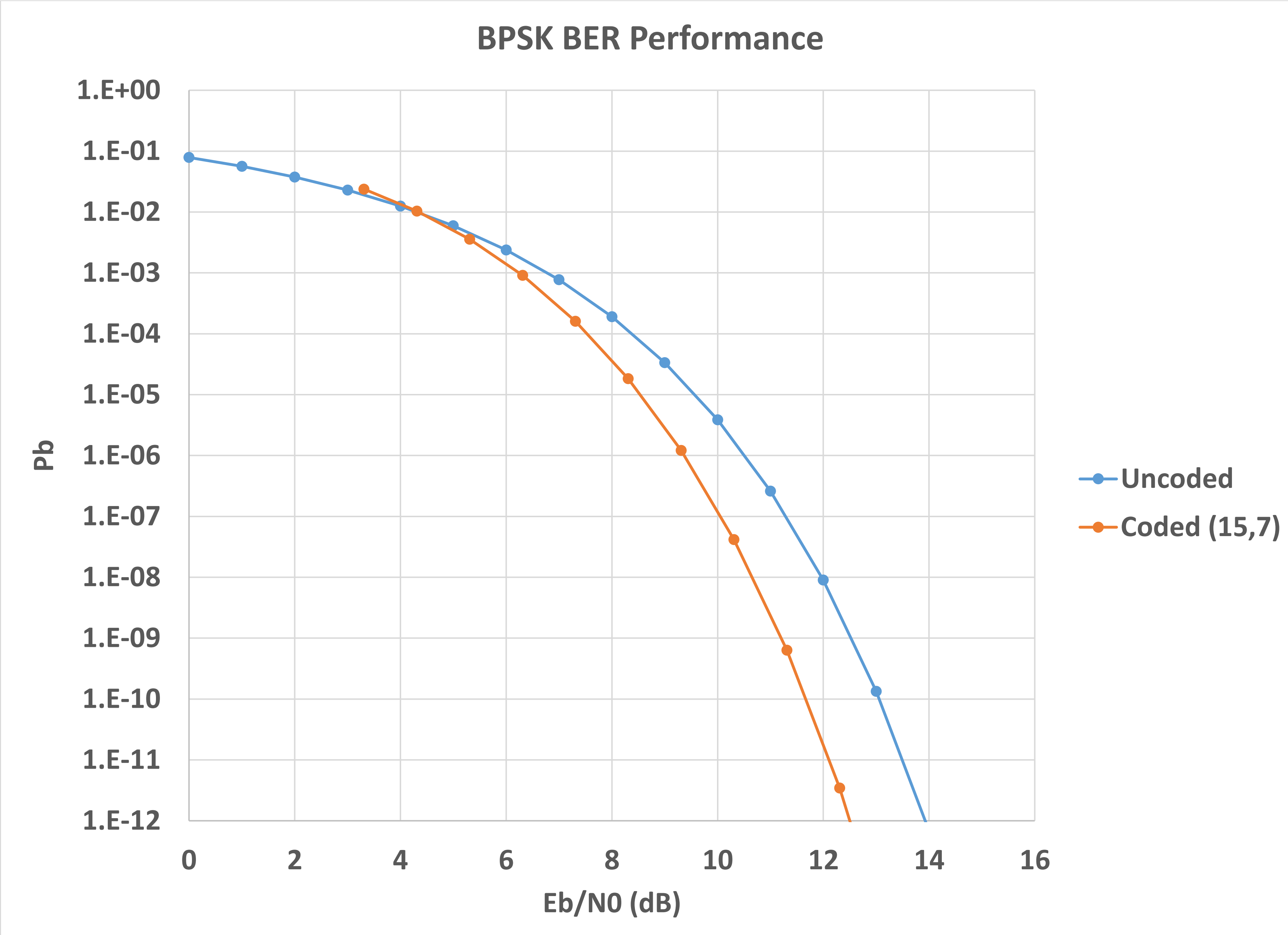
We must solve for c via

$$
\frac{1}{N}\sum_{n=0}^{N-1} Q\left(\frac{\text{Max} - \mu_{\text{X}}(\text{n})}{\sigma_X}\right) + \frac{1}{N}\sum_{n=0}^{N-1}\left[1 - Q\left(\frac{\text{Min} - \mu_{\text{X}}(\text{n})}{\sigma_X}\right)\right] = 0.01
$$

or

$$
\begin{aligned}
&\frac{1}{N}\sum_{n=0}^{N-1} Q\left(\frac{127.5 - cA_1\cos(2\pi n f_1/f_s) + cA_2\cos(2\pi n f_2/f_s)}{c\sigma}\right) \\
&+\frac{1}{N}\sum_{n=0}^{N-1}\left[1 - Q\left(\frac{-127.5 - cA_1\cos(2\pi n f_1/f_s) + cA_2\cos(2\pi n f_2/f_s)}{c\sigma}\right)\right] = 0.01
\end{aligned}
$$

where N varies over one period of the sampled waveform. Using $f_s = 15000$ Hz we find $N = 15$ and $c = 7.95$. Using $f_s = 150000$ Hz we find $N = 150$ and $c = 8.2$.


BPSK BER Performance
1.E+00
1.E-01
1.E-02
1.E-03
1.E-04
1.E-05
1.E-06
1.E-07
1.E-08
1.E-09
1.E-10
1.E-11
1.E-12
Pb
0
2
4
6
8
10
12
14
16
Eb/N0 (dB)
Uncoded
Coded (15,7)