EE 484

Homework 4 Solutions

Problem 1. Pulse Coded Modulation (PCM) is to be used to encode a signal. The signal ranges between the values -4 and +4. There are 3 bits or 8 levels (hence 8 code numbers) available. The levels assigned have symmetry like we demonstrated in class. The first three sample values obtained (before quantization) are 2.3, 0.4, and -1.6, respectively.

a. Find the quantized values for the three sample values.

Solution: We see that

$$\begin{aligned} 3.5 &\longleftrightarrow 111 \\ 2.5 &\longleftrightarrow 110 \\ 1.5 &\longleftrightarrow 101 \\ 0.5 &\longleftrightarrow 100 \\ -0.5 &\longleftrightarrow 011 \\ -1.5 &\longleftrightarrow 010 \\ -2.5 &\longleftrightarrow 001 \\ -3.5 &\longleftrightarrow 000 \end{aligned}$$

so

$$\begin{aligned} 2.3 &\rightarrow 2.5 \\ 0.4 &\rightarrow 0.5 \\ -1.6 &\rightarrow -1.5 \end{aligned}$$

b. Find the corresponding PCM sequences for the quantized values.

Solution: We see from above that the PCM sequence is 110 100 010.

Problem 2. Suppose we receive the analog signal

$$r_a(t) = A\cos(2\pi f t + \theta).$$

Here the amplitude A is a constant but we do *not* know its value. We do know that the frequency f is 100 Hz and the phase θ is $\pi/8$. We can follow the steps below to estimate the value of A. Assume for purposes of calculation that the value of A is 5, i.e., use $A = 5$ in the above signal in your calculations.

S1. Multiply $r_a(t)$ by $x(t)$, where $x(t) = \cos(2\pi 100t + \pi/8)$. Call the result $y(t)$.

S2. Integrate $y(t)$ from 0 to T and multiply the result by $2/T$. The result is your estimate of A.

a. Follow the 2 steps above and estimate A using $T = 6,\ 8,\ 16,\ 26,\ 126$ msec.

Solution.

$$y(t) = 2.5 + 2.5\cos\left(4\pi 100t + \pi/4\right).$$

$$\frac{2}{T}\int_0^T y(t)dt = 5 + \frac{5}{T}\frac{1}{4\pi 100}\left[\sin\left(4\pi 100T + \pi/4\right) - \sin\left(\pi/4\right)\right].$$

$$T = 6 \Rightarrow \hat{A} = 5.12$$

$$T = 8 \Rightarrow \hat{A} = 4.16$$

$$T = 16 \Rightarrow \hat{A} = 5.05$$

$$T = 26 \Rightarrow \hat{A} = 5.03$$

$$T = 126 \Rightarrow \hat{A} = 5.01$$

b. Explain why following the 2 steps above will give the exact answer for A as $T \to \infty$.

Solution. It is obvious from the above estimate that

$$\hat{A} = \frac{2}{T}\int_0^T y(t)dt \to A = 5$$

as $T \to \infty$.

c. Determine (analytically) the *finite* values of T that will make your estimate for A exact and using the smallest such T follow the two steps above again to estimate A.

Solution. We want

$$\sin(4\pi 100T + \pi/4) = \sin(\pi/4)$$

$$\Rightarrow 4\pi 100T = \pi/2,\ 2\pi,\ 5\pi/2,\ 4\pi, \ldots.$$

Thus,

$$4\pi 100T = (4k-3)\pi/2,\ k = 1, 2, 3, \ldots$$

or

$$4\pi 100T = 2k\pi,\ k = 1, 2, 3, \ldots.$$

So,

$$T = \frac{4k-3}{800},\ k = 1, 2, 3, \ldots$$

or

$$T = \frac{k}{200},\ k = 1, 2, 3, \ldots$$

Using $T = 1/800$ sec $= 1.25$ msec above gives $\hat{A} = 5$.

Problem 3. Suppose we receive the analog signal

$$r_a(t) = A\cos(2\pi 100t + \pi/8)$$

and sample it at 800 Hz to get the digital signal

$$r(n) = A\cos(0.25\pi n + \pi/8).$$

Here the amplitude A is a constant but we do *not* know its value. Furthermore, we do *not* know that the phase is $\pi/8$. We can follow the steps below to estimate the value of A. Assume for purposes of calculation that the value of A is 10, i.e., use $A = 10$ in the above signal in your calculations.

S1. Multiply $r(n)$ by $x_1(n)$ and $x_2(n)$, where $x_1(n) = \cos(0.25\pi n)$ and $x_2(n) = \sin(0.25\pi n)$. Call the results $y_1(n)$ and $y_2(n)$, respectively.

S2. Simply add up the values of $y_1(n)$ and $y_2(n)$ for $n = 0, 1, 2, \ldots N-1$ (some N) and take the average of each (divide by N) and then multiply the averages by 2. Call the results z_1 and z_2, respectively.

S3. Compute $\sqrt{z_1^2 + z_2^2}$. This is the estimate of A.

a. Follow the 3 steps above and estimate A using $N = 5, 11, 19$.

Solution.

$$y_1(n) = 2.5\cos(\pi/4) + 2.5\cos(0.5\pi n + \pi/4)$$

$$y_2(n) = -2.5\sin(\pi/4) + 2.5\sin(0.5\pi n + \pi/4)$$

$$z_1 = \frac{2}{N}\sum_{n=0}^{N-1} y_1(n), \quad z_2 = \frac{2}{N}\sum_{n=0}^{N-1} y_2(n).$$

$$\hat{A} = \sqrt{z_1^2 + z_2^2}.$$

$$N = 5 \Rightarrow \hat{A} = 5.10$$

$$N = 11 \Rightarrow \hat{A} = 5.02$$

$$N = 19 \Rightarrow \hat{A} = 5.01$$

b. Explain why following the 3 steps above will give the exact answer for A as $N \to \infty$.

Solution. Since sinusoids are periodic clearly we have

$$\lim_{N\to\infty} \frac{2}{N}\sum_{n=0}^{N-1} y_1(n) = 5\cos(\pi/4)$$

$$\lim_{N\to\infty} \frac{2}{N}\sum_{n=0}^{N-1} y_2(n) = -5\sin(\pi/4)$$

$$\Rightarrow \hat{A} = \sqrt{5^2\left(\cos^2(\pi/4) + \sin^2(\pi/4)\right)} = 5$$

as desired.

d. Determine (analytically) the *finite* values of N that will make your estimate for A exact and using the smallest such N follow the 3 steps above again to estimate A.

Solution. $y_1(n)$ and $y_1(n)$ have period 4 so using any multiple of 4 will give exact results. Using $N = 4$ above yields

$$\hat{A} = 5.$$

Problem 4. Suppose we receive the quantized digital signal

$$r_q(n) = \text{Round}\left[A\cos(0.25\pi n + \pi/8)\right]$$

where 'Round' means the samples are rounded to the nearest integer. The amplitude A is a constant but we do not know its value. Furthermore, we do not know that the phase is $\pi/8$. We can follow the steps below to estimate the value of A. [For purposes of calculation let A actually have the value 10.]

S1. Multiply $r_q(n)$ by $x_1(n)$ and $x_2(n)$, where $x_1(n) = \cos(0.25\pi n)$ and $x_2(n) = \sin(0.25\pi n)$. Call the results $y_1(n)$ and $y_2(n)$, respectively.

S2. Simply add up the values of $y_1(n)$ and $y_2(n)$ for $n = 0, 1, 2, \ldots N-1$ (some N) and take the average of each (divide by N) and then multiply the averages by 2. Call the results z_1 and z_2, respectively.

S3. Compute $\sqrt{z_1^2 + z_2^2}$. This is the estimate of A.

a. Follow the 3 steps above and estimate A using $N = 4$.

Solution.

$$\hat{A} = 9.85.$$

b. Repeat (a) for $N = 8$.

Solution.

$$\hat{A} = 9.85.$$

c. Based on your answers to parts (a) and (b) what would your estimate for A be if N is 4000. Explain why the estimate is not becoming exact even for very large N.

Solution.

$$\hat{A} = 9.85.$$

The error introduced by the rounding operation repeats every period of the signal and we are selecting the same samples each period. Thus, the errors do not average out to zero in this case.