EE 484

Homework 2 Solutions

Problem 1. Let $x(n)$ be an independent and identically distributed random sequence with each $x(n)$ having mean 0 and variance σ^2. Suppose we form

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

where,

$$h(n) = \begin{cases} (-\alpha)^{n/2}, & n \geq 0, \text{ even} \\ 0, & \text{elsewhere} \end{cases}$$

where $0 < \alpha < 1$. Find $S_y(f)$, the power spectral density of Y.

Solution:

$$S_y(f) = |H(f)|^2 S_x(f)$$

where $S_x(f) = \sigma^2$ and

$$\begin{aligned} H(z) &= \sum_{n=0}^{\infty} h(n)z^{-n} = \sum_{n=0}^{\infty} h(2n)z^{-2n} \\ &= \sum_{n=0}^{\infty} (-\alpha z^{-2})^n = \frac{1}{1+\alpha z^{-2}} \end{aligned}$$

Problem 2. Consider a real Gaussian random sequence $x(n)$, n an integer, with

$$E[x(n)] = 0, \quad E\left[x(n)^2\right] = 1, \quad E[x(n)x(m)] = \rho^{|n-m|}$$

where $0 < \rho < 1$. Let

$$y(n) = \begin{cases} \dfrac{x(n)}{n}, & n \neq 0 \\ 0, & n = 0. \end{cases}$$

a. Is $x(n)$ wide sense stationary?

Solution. Yes, $x(n)$ is WSS from its definition.

b. Find the covariance of $y(n)$ and state whether or not it is wide sense stationary.

Solution. $E[y(n)] = 0$ so

$$K_y(n,m) = E\left[\frac{x(n)}{n}\frac{x(m)}{m}\right] = \frac{\rho^{|n-m|}}{nm}$$

which is a function of n and m so is not WSS.

Problem 3. A zero mean sequence of i.i.d. random variables $x(n)$ form the input to a causal linear system defined by the linear difference equation

$$y(n) = \alpha y(n-1) + x(n), \quad |\alpha| < 1.$$

Find the impulse response $h(n)$ of a second system such that the sequence

$$z(n) = \sum_{i=0}^{k} h(i)y(n-i)$$

has a constant power spectral density, i.e.,

$$S_Z(f) = \sigma_z^2, \quad f \in \left[-\tfrac{1}{2},\ \tfrac{1}{2}\right).$$

Solution: Assume that $x(n)$ has variance σ_X^2 so that $S_X(f) = \sigma_X^2$. Note that

$$y(n) = \sum_{n=0}^{\infty} \alpha^i x(n-i)$$

so $Y(z) = G(z)X(z)$ where

$$G(z) = \frac{1}{1-\alpha z^{-1}}.$$

We want to find $H(z)$ such that $H(z)Y(z) = \sigma_Z$ so that $Z_Z(f) = \sigma_Z^2$. Thus, we want $H(z)G(z)X(z) = \sigma_Z$. We find

$$H(z) = \frac{\sigma_Z}{G(z)X(z)} = \frac{\sigma_Z}{\sigma_X}(1-\alpha^{-1})$$

which yields the IIR filter

$$h(n) = \frac{\sigma_Z}{\sigma_X}\left(\delta(n) - \delta(n-1)\right).$$