EE 484

Homework 1 Solutions

Problem 1. Determine for each of the following whether or not the discrete-time system is linear, and/or time-invariant.

a. $y(n) = 2\sin[x(n)]$.

   Solution: not linear

b. $y(n) = x(n-2)$.

   Solution: linear

c. $y(n) = e^{|x(n)|}$.

   Solution: not linear

d. $y(n) = nx(n)$.

   Solution: linear

Problem 2. Compute the Fourier transform of the following

$$x(t) = \begin{cases} t, & 0 \le t \le 1, \\ 2-t, & 1 < t \le 2, \\ 0, & elsewhere \end{cases}$$

and sketch a plot of its magnitude in the frequency domain.

Solution: If we define a rectangle function as having the value 1 for $-1/2 \le t \le 1/2$ and 0 elsewhere, then $x(t)$ is the convolution of a shifted version of the rectangle function where the shift value is 1/2. Now the rectangle function has a Fourier transform of $\text{sinc}(\pi f)$. Hence, after accounting for the shift we have

$$X(f) = \left[e^{-j\pi f}\text{sinc}(\pi f)\right]^2 = e^{-j2\pi f}\text{sinc}^2(\pi f).$$

Problem 3. Show that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1.$$

Solution: It is enough to show

$$I^2 = \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \right)^2 = 1.$$

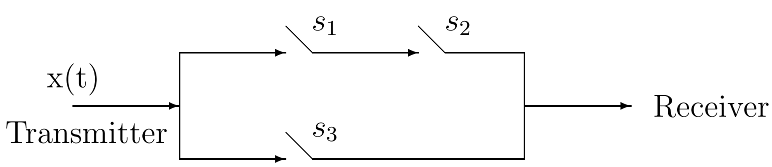
Now

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} dx dy. \end{aligned}$$

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then,

$$\begin{aligned} I^2 &= \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2/2} r dr d\theta \\ &= 1. \end{aligned}$$

Problem 4. Consider the transmission of a signal as shown in the following diagram.



A signal is transmitted along two paths as shown. In the upper path there are two switches to pass through while in the lower path there is one switch to pass through. Each switch s_i operates independently and allows the signal to pass with probability p_i for $i = 1, 2, 3$. The signal transmission is successful if the signal $x(t)$ sent at the transmitter reaches the receiver along either or both paths. Find the probability that the transmission is successful if

a. $p_1 = 9/10,\ p_2 = 1/2,\ p_3 = 3/4.$

Solution: Let U be the event of the signal passing thru the upper branch and let L be the event of it passing thru the lower branch. Then,

$$P(success) = P(U)+P(L)-P(U \cap L) = \frac{9}{10}\frac{1}{2}+\frac{3}{4}-\frac{9}{10}\frac{1}{2}\frac{3}{4} = \frac{73}{80} = 0.9125.$$

b. $p_1 = 9/10,\ p_2 = 1/2,\ p_3$ is a random variable uniformly distributed in the interval (0, 1).

Solution: Conditioned on p

$$P(success|p) = \frac{9}{10}\frac{1}{2} + p - \frac{9}{10}\frac{1}{2}p = \frac{9}{20} + p - \frac{9}{20}p.$$

So,

$$P(success) = \int_0^1 \left(\frac{9}{20} + p - \frac{9}{20}p\right) dp = 0.725.$$

Problem 5. Suppose the random variable X has mean 1 and variance 4. Let $Y = X^2$. Find the mean of Y.

Solution:

$$E[Y] = E[X^2] = Var(X) + (E[X])^2 = 4 + 1 = 5.$$

Problem 6. Suppose the normal random variable X has mean 4 and variance 8. Let $Y = X^2$. Find the variance of Y.

Solution:

$$E[Y] = E[X^2] = Var(X) + (E[X])^2 = 8 + 16 = 24.$$

$$E[Y^2] = E[X^4].$$

First let

$$Z = \frac{X - E[X]}{\sqrt{Var(X)}} = \frac{X - \mu_X}{\sigma_X}.$$

Then Z is a standard normal random variable with mean 0 and variance 1. Note that $E[Z^4] = 3$. Now

$$X = \mu_X + \sigma_X Z$$

so

$$X^4 = \mu_X^4 + 4\mu_X^3\sigma_X Z + 6\mu_X^2\sigma_X^2 Z^2 + 4\mu_X\sigma_X^3 Z^3 + \sigma_X^4 Z^4.$$

Hence,

$$E[X^4] = \mu_X^4 + 6\mu_X^2\sigma_X^2 + 3\sigma_X^4$$

or

$$E[X^4] = 256 + 768 + 192 = 1216.$$

Thus,

$$Var(Y) = E[Y^2] - (E[Y])^2 = E[X^4] - (E[X^2])^2 = 1216 - 576 = 640.$$