13.0 Random Processes

13.1 Introduction

Definition: A random process is a set of indexed random variables $X(u,t)$ defined on (U,T,P) where t takes values in some index set T.

For any fixed $t = t_0 \in T, X(u,t_0)$ is a random variable.
For a fixed $u = u_0 \in U, X(u_0,t)$ is a sample function.

If T is finite, we have a random vector.
If T is countable, we have a random sequence.
If $T = \mathbf{R}$, we have a random process.
If $T = \mathbf{R}^n$, we have a random field.
The case $T = \mathbf{R}^2$ is used in image processing.

We often write $X(t)$ for $X(u,t)$

Characterization of Random Process:
Random Variable: $F_X(x) = P(X \leq x)$
First order distribution and density of a random process,

$$F_X(u,t) = P(X(u,t) \leq x),$$

$$f_X(u,t) = \frac{dF_X(u,t)}{dx}.$$

In general, random variables for different $t \in T$ are neither independently nor identically distributed, so 1^{st} order pdf does not characterize the random process.

N^{th} order distribution and pdf:

$$F_X(x_1,\cdots,x_n;t_1,\cdots,t_n) = P(X(u,t_1) \leq x_1 \cdots, X(u,t_n) \leq x_n)$$

leads to $F_X(x_1,\cdots,x_n;t_1,\cdots,t_n)$ which contains all information available. This is usually too complicated to work with. Instead we rely on 1^{st} and 2^{nd} order statistics. Note that these completely characterize the Gaussian case and is often good enough for other distributions.

13.2 The Second Moment Theory of Random Processes

Mean:

$$\mu_X(t) = E[X(u,t)] \qquad \forall\, t \in T$$
$$= \int_{-\infty}^{\infty} x f_X(x,t)\, dx$$

Correlation:

$$R_X(t_1,t_2) = E[X(u,t_1)X^*(u,t_2)] \quad \forall\, t_1, t_2 \in T.$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1,x_2;t_1,t_2)\, dx_1 dx_2$$

Covariance:

$$K_X(t_1,t_2) = E[(X(u,t_1) - \mu_X(t_1))(X(u,t_2) - \mu_X(t_2))^*]$$
$$K_X(t_1,t_2) = R_X(t_1,t_2) - \mu_X(t_1)\mu_X^*(t_2)$$

13.3 Examples of Random Processes

1)

$$X(u,t) = A(u) \quad \forall \mu \in U, t \in T$$

$A(u)$ is a random variable with mean m and variance σ^2

$$\mu_X(t) = E[A(u)] = m \quad \textit{(not dependent on t)}$$
$$R_X(t_1,t_2) = E[X(u,t_1)X^*(u,t_2)] = E[A(u)A(u)] = \sigma^2 + m^2$$
$$K_X(t_1,t_2) = \sigma^2$$

<u>Note</u> When we compute $\mu_X(t)$ as $E[X(u,t)]$, then we are in effect computing the ensemble average for each t. Similarly for R_X and K_X.

Say, $Z(u) \sim N(0, \sigma^2)$. $X(u,t) = Z(u)$

$$\mu_X(t) = 0, K_X(t_1, t_2) = \sigma^2$$

Let us observe this $X(u,t)$ over time t. $X(u,t)$ does not change over time. We just observe some constant sample and if we do this many times on the average the constant will be 0 but any particular outcome, i.e., $X(u_0, t)$ is some constant $Z(u_0)$.

So, time average of $X(u_0, t)$ is a constant and not necessarily equal to the ensemble average for some $X(u, t_0)$.

If time average equals to ensemble average, we have an ergodic process.

2)

$$X(u,t) = \sin(t - \phi(u))$$
$$\phi(u) \sim U(-\pi, \pi)$$

So,

$$f_\phi(\phi) = \begin{cases} \dfrac{1}{2\pi} & |\phi| < \pi, \\ 0, & \text{elsewhere} \end{cases}$$

$$\begin{aligned} \mu_X(t) &= E[\sin(t - \phi(u))] \\ &= \int_{-\pi}^{\pi} \frac{1}{2\pi} \sin(t - \phi) d\phi \\ &= \frac{1}{2\pi} \cos(t - \phi)|_{-\pi}^{\pi} = 0 \end{aligned}$$

$$K_X(t_1, t_2) = R_X(t_1, t_2)$$
$$E[\sin(t_1 - \phi) \sin(t_2 - \phi)]$$
$$= \frac{1}{2} E[\cos(t_1 - t_2) - \cos(t_1 + t_2 - 2\phi)]$$
$$= \frac{1}{2} \cos(t_1 - t_2) - \frac{1}{4\pi} \underbrace{\int_{-\pi}^{\pi} \cos(t_1 + t_2 - 2\phi) d\phi}_{0}$$

$$K_X(t_1, t_2) = \frac{1}{2}\cos(t_1 - t_2)$$

Note $K_X(t_1, t_2)$ is a function of $(t_1 - t_2)$ only. So,

$$K_X(t_1 + \tau, t_2 + \tau) = \frac{1}{2}\cos(t_1 - t_2)$$

This is 2^{nd} order stationarity.

13.4 Properties of Correlation Functions

1) $\mu_X(t)$ is any real function defined on T.
2) $R_X(t_1, t_2) = R_X^*(t_2, t_1)$
3) $R_X(t_1, t_2)$ is a non-negative definite function.
4) $R_X(t, t) \geq 0 \quad \forall t \in T$.
5) $|R_X(t_1, t_2)| \leq \sqrt{R_X(t_1, t_1)}\sqrt{R_X(t_2, t_2)}$

1),2),3) are necessary and sufficient condition for the existence of a random process with mean $\mu_X(t)$ and correlation $R_X(t_1, t_2)$.

14.0 LTI Systems

14.1 Definitions

Here let H be a mapping

$$H : L_1 \rightarrow L_2$$

where, L_1 and L_2 are two linear spaces.

Definition: H is said to be a linear system if

$$H(ax) = aH(x)$$

$$H(x_1 + x_2) = H(x_1) + H(x_2)$$

for scalar a and $x_1, x_2 \in L_1$.

Translation (Shift) Operations

Let T be an Abelian (commutative) group with binary operation "+", i.e.,

1. $t_1, t_2 \in T \Rightarrow t_1 + t_2 \in T$
2. $t_1, t_2, t_3 \in T \Rightarrow t_1 + (t_2 + t_3) = (t_1 + t_2) + t_3)$
3. $\exists\ 0 \in T$ such that $t + 0 = 0 + t = t\ \forall\ t \in T$
4. For every $t \in T\ \exists\ t^{-1} \in T$ such that $t + t^{-1} = 0\ [t^{-1} = -t]$
5. (Abelian) For every $t_1, t_2 \in T,\ t_1 + t_2 = t_2 + t_1$

Definition: The shift operator T_τ is defined as

$$T_\tau(x)(t) = x(t + \tau)$$

for $t \in T$, "+" is the binary operation. Here T is an Abelian group.

Note: $T_{\tau_1}(T_{\tau_2(x)}) = T_{\tau_1+\tau_2}(x)$.

Definition: A linear system H is said to be *time invariant* or *shift invariant* if

$$H(T_\tau(x)) = T_\tau(H(x))$$

i.e., H commutes with T_τ.

Eigenfunctions
If H is an LTI (or LSI) system then the functions

$$e_f(t) = e^{i2\pi ft},\ \forall\ t \in T$$

are the system *eigenfunctions*, i.e.,

$$He_f = H(f)e_f$$

where,

i. $f \in \{0,\ 1/n,\ 2/n, \ldots,\ (n-1)/n\}$ for $T = [0, 1, 2, \ldots, n-1]$

ii. $f \in [-1/2,\ 1/2]$ for $T = \mathbf{Z}$

iii. $f \in [0,\ \pm 1/T,\ \pm 2/T, \ldots]$ for $T = [0, A)$

iv. $f \in [-\infty,\ \infty]$ for $T = \mathbf{R}$

14.2 Discrete Time Systems

14.2.1 Eigensequences

Let H be a discrete time invariant linear system with $T = \mathbf{Z}$, so $t \in \{0, \pm 1, \pm 2, \ldots\}$. Let

$$e_f(n) = e^{i2\pi fn}.$$

We will show

$$He_f = H(f)e_f$$

where

$$H(f) = \sum_k h(k)e^{-i2\pi fk}.$$

Now

$$He_f = H(e_f)(n) = h(n) * e_f(n)$$

$$= \sum_k h(k)e^{i2\pi f(n-k)} = e^{i2\pi fn} \sum_k h(k)e^{-i2\pi fk} = e^{i2\pi fn}H(f) = H(f)e_f.$$

Now consider $x(n) = e^{i2\pi fn}$. If $x(n)$ is operated on by H then the output $y(n)$ is

$$y(n) = e^{i2\pi fn}H(f)$$

where $H(f)$ is a constant for a fixed f. Now let D^{-k} denote a delay of the input by k samples. Then

$$D^{-k}H\left\{x(n)\right\} = HD^{-k}\left\{x(n)\right\}.$$

Define the impulse response

$$h(n) = H\left\{\delta(n)\right\}$$

where $\delta(n)$ is the delta function that has the value 1 at $n = 0$ and is 0 otherwise. Then

$$x(n) = \sum_k x(k)\delta(n-k)$$

which is a weighted sum of impulses. Thus,

$$h(n) = H\left\{x(n)\right\} = H\left\{\sum_k x(k)\delta(n-k)\right\} = \sum_k x(k)H\left\{\delta(n-k)\right\}.$$

Now

$$H\left\{\delta(n-k)\right\} = HD^{-k}\left\{\delta(n)\right\} = D^{-k}H\left\{\delta(n)\right\} = D^{-k}h(n) = h(n-k).$$

So

$$y(n) = \sum_k x(k)h(n-k) = \sum_k h(k)x(n-k).$$

Now let $x(n)$ be the eigensequence $e^{i2\pi fn}$. Then

$$y(n) = H(f)x(n) = x(n)H(f) = e^{i2\pi fn}\sum_k h(k)e^{-i2\pi fn}.$$

Note

$$H(f) = \sum_k h(k)e^{-i2\pi fn}$$

is an eigenvalue for a fixed f. $H(f)$ is a Fourier series and denotes the frequency response. Thus, $h(k)$ are the Fourier series coefficients of $H(f)$, i.e.,

$$h(k) = \int_{-1/2}^{1/2} H(f)e^{i2\pi fk}df.$$

Note that $H(f)$ is periodic with period $1 = 1/2 - (-1/2)$. Also, $H(f)$ exists for systems for which $\sum_k |h(k)| < \infty$.

14.2.2 Fourier Analysis

Note that if $y(n) = x(n) * h(n)$ then $Y(f) = X(f)H(f)$.

Now consider the case

$$y(n) = \sum_{k=0}^{M} b_k x(n-k) - \sum_{k=1}^{N} a_k y(n-k).$$

Then

$$Y(f) = X(f) \sum_{k=0}^{M} b_k e^{-i2\pi fk} - Y(f) \sum_{i=1}^{N} a_k e^{-i2\pi fk}.$$

So

$$H(f) = \frac{Y(f)}{X(f)} = \frac{\sum_{k=0}^{M} b_k e^{-i2\pi fk}}{1 + \sum_{i=1}^{N} a_k e^{-i2\pi fk}}$$

which is the frequency response of the system.

15.0 Wide Sense Stationary (WSS) Random Processes

15.1 Definitions

Definition: A random process $X(u,t)$ is stationary if

$$F(\mathcal{T}_\tau(\underline{x})) = F(\underline{x})$$

Definition: A random process is wide sense stationary if it has time invariant 1^{st} and 2^{nd} order statistics.

So, 1) $\mu_X(t) = \mu_X(t+\tau)$ Let $\tau = -t$

$$\mu_X(t) = \mu_X(0) \longrightarrow \quad \textit{some constant}$$

2) $R_X(t_1,t_2) = R_X(t_1+\tau, t_2+\tau)$ Let $\tau = -t_2$

$$R_X(t_1,t_2) = R_X(t_1-t_2, 0) = R_X(t_1-t_2)$$

15.2 Power Spectral Density (PSD) in Discrete Time Systems

Positive Semi-Definite Property

$$\sum_{k=-N}^{N}\sum_{\ell=-N}^{N} a_k R_X(k,\ell) a_\ell^* \geq 0 \quad \forall N \in \mathcal{T} \quad \textit{(discrete)}$$

Let $a_k = e^{-i2\pi fk}$. Assume $X(u,n)$ is WSS.

$$\sum_{k=-N}^{N}\sum_{\ell=-N}^{N} e^{-i2\pi f(k-\ell)} R_X(k-\ell) \geq 0$$

Let $m = k-\ell$; $n = k+\ell$

$$\Longrightarrow \sum_{m=-2N}^{2N} (2N+1-|m|)e^{-i2\pi fm} R_X(m) \geq 0 \quad \textit{(think)}$$

divide by $(2N+1)$, take limit,

$$\lim_{N\to\infty} \sum_{m=-2N}^{2N} (1 - \frac{|m|}{2N+1}) e^{-i2\pi fm} R_X(m) \geq 0$$

Define

$$\underbrace{S_X(f) = \sum_{m=-\infty}^{\infty} R_X(m) e^{-i2\pi fm}}_{\textbf{Power Spectral Density}}, \qquad f \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

So, PSD = Fourier Transform of autocorrelation function.

Properties

1) $R_X(m)$ are Fourier Series coefficient for $S_X(f)$.

$$R_X(m) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f) e^{i2\pi fm} df$$

$$R_X(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f) df$$

$$R_X(0) = E[|X(u)|^2]$$

2) Periodic: $S_X(f) = S_X(f+k)$ for integer k.

3) Any P.S.D. correlation has a non-negative PSD.

4) $S_X(f)$ is real.

5) If $R_X(u)$ is real, then $S_X(f) = S_X(-f)$.

16.0 Stochastic Inputs to LSI Systems

16.1 General Systems

$$y(u,n)=\sum_{m=0}^{\infty} h(m,n)x(u,m)$$

$h(m,n)$ is not shift-invariant in general and $y(u,n)$ does not exist in general (there may be some sample processes that are not summable).

$y(u,n)$ exists in the mean square sense if

1) BIBO
2) Variances are bounded

Let $y_k(u,n)=\sum_{m=0}^{k} h(m,n)x(u,m)$. If for every $\epsilon > 0 \quad \exists$ a number $N(\epsilon)$ $s.t.$

$$E[|y_k(u,n)-y_\ell(u,n)|^2] < \epsilon \quad \forall k,\ell > N(\epsilon)$$

then

$$y_k(u,n) \xrightarrow{m.s.} y(u,n)$$

Consider WLOG that $k > \ell$,

$$E\left[|y_k(n)-y_\ell(n)|^2\right] = E\left[\left|\sum_{m=\ell+1}^{k} h(m,n)x(m)\right|^2\right]$$

$$=\sum_{m=\ell+1}^{k}\sum_{p=\ell+1}^{k} h(m,n)h^*(n,p)R_X(n,p)$$

$$\leq \sum_{m=\ell+1}^{k}\sum_{p=\ell+1}^{k} |h(m,n)||h^*(n,p)||R_X(n,p)|$$

$$\leq [\sum_{p=\ell+1}^{k} |h(n,p)|(R_X(p,p))^{\frac{1}{2}}]^2$$

If

$$\lim_{k\to\infty}\sum_{p=0}^{k} |h(n,p)|(R_X(p,p))^{\frac{1}{2}}$$

exists, then as $k, \ell \longrightarrow \infty$

$$\sum_{p=\ell+1}^{k} |h(n,p)|(R_X(p,p))^{\frac{1}{2}} \longrightarrow 0$$

Theorem: If $X(u,n)$ is a sequence of random variables with $R_X(n,n) < \infty \quad \forall n$ and $h(n,m)$ is absolutely summable, i.e.,

$$\sum_{m=0}^{\infty} |h(m,n)| < \infty \quad \forall n$$

then,

$$\sum_{m=0}^{\infty} |h(n,m)| x(u,m)$$

exists in the mean square sense.

16.2 WSS in LSI Systems

Assume $h(n)$ is causal, LSI, BIBO stable. Recall we have BIBO stability and causality *if and only if* $\sum_{n=0}^{\infty} |h(n)| < \infty$, and $h(n) = 0$ for $n < 0$. Assume $x(n) is\ WSS$ and $E[x(n)^2] < \infty$.

Mean

$$E[y(n)] = E[h(n) * x(n)]$$
$$= \sum_{k=0}^{\infty} h(k) E[x(n-k)]$$
$$\mu_y = \mu_x \sum_{k=0}^{\infty} h(k) \quad \textit{(constant)}$$

Cross Correlation

$$R_{XY}(n,m) = E[x(u) y^*(m)]$$
$$= E[x(u) \sum_{k=0}^{\infty} h^*(k) x^*(m-k)]$$
$$= \sum_{k=0}^{\infty} h^*(k) R_X(n, m-k)$$
$$= \sum_{k} h^*(k) R_X(n-m+k)$$

Let $\ell = n - m$

$$R_{XY}(\ell) = \sum_{k=0}^{\infty} h^*(k) R_X(\ell + k)$$

$$= R_X(\ell) * h^*(-\ell)$$

Similar derivation for cross covariance, $K_{XY}(n, m)$

<u>Correlation of Y</u>

$$R_Y(n, m) = E[y(n)y^*(m)]$$

$$= E[\sum_p h(p)x(n-p)y^*(m)]$$

$$= \sum_{p=0}^{\infty} h(p) \underbrace{E[x(n-p)y^*(m)]}_{R_{XY}(n-p-m)}$$

$$= \sum_{p=0}^{\infty} h(p) \sum_{k=0}^{\infty} h^*(k) R_X(n - p - m + k)$$

Let $q = n - m$

$$R_Y(q) = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} h(p) h^*(k) R_X(q - p + k)$$

$$= h(q) * R_X(q) * h^*(-q)$$

Recall

$R_Y = H R_X H^\dagger$ *(note similarity between time domain and freq. domain).*

Similar derivation for covariance of $Y, K_Y(n, m)$.

Fourier Space

Define a cross-spectral density as

$$S_{XY}(f) = \sum_{n=-\infty}^{\infty} R_{XY}(n) e^{-i2\pi f n}, f \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} h^*(k) R_X(n+k) e^{-i2\pi f(n+k)} e^{i2\pi f k}$$

or

$$S_{XY}(f) = S_X(f)H^*(f).$$

Now,

$$\begin{aligned} S_Y(f) &= \sum_{n=-\infty}^{\infty} R_Y(n)e^{-i2\pi fn} \\ &= H(f)H^*(f)S_X(f). \end{aligned}$$

$$S_Y(f) = |H(f)|^2 S_X(f).$$

Recall, $R_Y(q) = h(q) * h^*(-q) * R_X(q)$

Ex1 Suppose $x(n)$ is an i.i.d., zero-mean sequence, var $= \sigma_X^2$.

$$K_X(n) = \sigma_X^2 \delta(n)$$

$$\begin{aligned} S_X(f) &= \sum_{n=-\infty}^{\infty} K_X(n)e^{-i2\pi fn} \\ &= \sigma_X^2 \quad \longrightarrow \quad \textbf{"White Noise"} \end{aligned}$$

(power equally spread over all frequences)

Consider the moving average

$$y(n) = \sum_{k=0}^{M} b_k x(n-k).$$

Then

$$H(f) = \sum_{k=0}^{M} b_k e^{-i2\pi fk}.$$

Thus

$$S_Y(f) = \left|\sum_{k=0}^{M} b_k e^{-i2\pi fk}\right|^2 \sigma_X^2.$$

Now consider the special case $M = 1,\ b_0 = h(0) = 1,\ b_1 = h(1) = -1$ and $h(n) = 0$ for $n \neq 0, 1$. Then

$$y(n) = x(n) - x(n-1).$$

So

$$H(f) = \sum_{k=0}^{1} h(k)e^{-i2\pi fk} = 1 - e^{-i2\pi f}.$$

$$|H(f)|^2 = 2 - 2\cos(2\pi f)$$

and

$$S_Y(f) = \sigma_X^2(2 - 2\cos(2\pi f)).$$

17.0 Spectral Concepts

17.1 Spectral Densities

$$S_X(f) = \sum_{k=-\infty}^{\infty} R_X(k)e^{-i2\pi fk} = \sum_{k=-\infty}^{\infty} K_X(k)e^{-i2\pi fk} + m_X^2\delta(f)$$

where $\delta(f)$ is a generalized function defined by its sifting property

$$\int_{-\infty}^{\infty} x(f)\delta(f-f_0)df = x(f_0)$$

provided $x(f)$ is a function continuous at $f = f_0$. So

$$R_X(n) = K_X(n) + \int_{-1/2}^{1/2} m_X^2\delta(f)e^{i2\pi fn}df$$

$$= K_X(n) + m_X^2.$$

In general, if

$$R_X(n) = K_X(n) + \sum_k a_k e^{-i2\pi f_k}$$

then

$$S_X(f) = \tilde{S}_X(f) + \sum_k a_k\delta(f-f_k)$$

where

$$K_X(n) \longleftrightarrow \tilde{S}_X(f).$$

17.2 Spectral Factorization

System Function

Consider

$$H(z) = \sum_n h(n)z^{-n}.$$

The region of convergence (ROC) of this z-transform is

$$ROC = \left\{z : \sum_n |h(n)z^{-n}| < \infty\right\}.$$

This guarantees uniform convergence.

Causal Sequences

Here $h(n) = 0$ for $n < 0$. Then

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots.$$

Note that $h(n)$ is causal if and only if $H(z)$ converges as $|z| \to \infty$.

Stable Causal Sequences

$h(n)$ is causal and stable if

$$\sum_{n=0}^{\infty} |h(n)| < \infty.$$

This is equivalent to

$$\sum_{n=0}^{\infty} |h(n)z^{-n}|_{|z|=1} < \infty.$$

Thus causal $h(n)$ is stable if and only if $H(z)$ converges on the unit circle in the z-plane.

Poles

Here we identify those values of z that make $H(z) \to \infty$.

Example:

$$h(n) = \alpha^n u(n), \ |\alpha| < 1.$$

$$H(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1 - \alpha z^{-1}}.$$

To find the pole we set $(1 - \alpha z^{-1}) = 0$ to get $\alpha = z$.

Zeros

Here we identify those values of z that make $H(z) = 0$.

Example:

$$h(n) = \delta(n) - \alpha\delta(n-1).$$

$$H(z) = (1 - \alpha z^{-1}.)$$

To find the zero we set $(1 - \alpha z^{-1}) = 0$ to get $\alpha = z$.

Poles and ROC

ROC for causal sequences= $\{z : d_{max} < |z| < \infty\}$ where d_{max} is the magnitude of the largest pole.

Poles and Stability

Since ROC must include the unit circle all poles must lie inside the unit circle.

Rational System Function

Consider the linear difference equation

$$y(n) = \sum_{k=0}^{M} b_k x(n-k) - \sum_{k=1}^{N} a_k y(n-k).$$

Take z-transform

$$Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) - \sum_{k=1}^{N} a_k z^{-k} Y(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=1}^{N} a_k z^{-k}}.$$

Poles occur at roots of

$$\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right) = d_j,\ \ j = 1, 2, \ldots, N.$$

Zeros occur at roots of

$$\left(\sum_{k=0}^{M} b_k z^{-k}\right) = c_j,\ \ j = 1, 2, \ldots, M.$$

$$S_Y(f) = |H(f)|^2 S_X(f) = H(f)H^*(f)S_X(f)$$
$$= H(z)H^*(z)S_X(f)\Big|_{z=e^{i2\pi f}}.$$

If $h(n)$ is real then $H(z) = H^*(z^*) \Rightarrow H^*(z) = H(z^*)$. Now

$$H^*(z)\Big|_{z=e^{i2\pi f}} = H^*\left(e^{i2\pi f}\right) = H\left(e^{-i2\pi f}\right) = H(z^{-1})\Big|_{z=e^{i2\pi f}}.$$

So for $h(n)$ real

$$S_Y(f) = H(z)H(z^{-1})S_X(f)\Big|_{z=e^{i2\pi f}}.$$

Example: Let $x(n)$ be an i.i.d. sequence with mean zero and variance $\sigma^2 = 1$. $x(n)$ is applied to a filter with z-transform $H(z)$. The output is $y(n)$. Say

$$K_Y(m,n) = \alpha^{|m-n|},\ |\alpha| < 1 \text{ and is real.}$$

Find $H(z)$.

Let $k = m - n$. Then

$$K_Y(k) = \alpha^{|k|}.$$
$$S_Y(f) = \sum_{k=-\infty}^{\infty} \alpha^{|k|} e^{-i2\pi fk}$$
$$= \sum_{k=0}^{\infty} \alpha^k e^{-i2\pi fk} + \sum_{k=-\infty}^{0} \alpha^{-k} e^{-i2\pi fk} - 1$$
$$= \sum_{k=0}^{\infty} \left(\alpha e^{-i2\pi f}\right)^k + \sum_{k=0}^{\infty} \left(\alpha e^{i2\pi f}\right)^k - 1$$
$$= \frac{1}{1-\alpha e^{-i2\pi f}} + \frac{1}{1-\alpha e^{i2\pi f}} - 1$$
$$= \frac{1-\alpha^2}{\left(1-\alpha e^{-i2\pi f}\right)\left(1-\alpha e^{i2\pi f}\right)}.$$

Replace $e^{i2\pi f}$ by z. Then

$$S_Y(f) = \frac{1-\alpha^2}{(1-\alpha z^{-1})(1-\alpha z)}.$$

We have poles at $z = \alpha$ and $z = 1/\alpha$. We can write

$$S_Y(z) = H(z)H(z^{-1})$$

where

$$H(z) = \frac{(1-\alpha^2)^{1/2}}{1-\alpha z^{-1}}.$$