9.0 Fast Fourier Transform

The Fast Fourier Transform (FFT) is an efficient method for computing the DFT.

$N-$point Decimation-in-Time (DIT) FFT

Recall,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad W_N = e^{-i2\pi/N}.$$

We may write this as

$$X(k) = \sum_{\substack{n=0 \\ n \ even}}^{N-1} x(n) W_N^{kn} + \sum_{\substack{n=0 \\ n \ odd}}^{N-1} x(n) W_N^{kn}.$$

Let us now assume N is even (we can change things slightly if this is not the case or even easier we can make N even by zero padding, if necessary). Then,

$$X(k) = \sum_{m=0}^{\frac{N}{2}-1} x(2m) W_N^{k2m} + \sum_{m=0}^{\frac{N}{2}-1} x(2m+1) W_N^{k(2m+1)}.$$

Now,

$$W_N^{k2m} = e^{-i2\pi k2m/N} = e^{-i2\pi km \big/ N/2} = W_{N/2}^{km}.$$

Thus,

$$X(k) = \sum_{m=0}^{\frac{N}{2}-1} x(2m) W_{N/2}^{km} + W_N^k \sum_{m=0}^{\frac{N}{2}-1} x(2m+1) W_{N/2}^{km}.$$

Define

$$G(k) = \sum_{m=0}^{\frac{N}{2}-1} x(2m) W_{N/2}^{km}$$

and

$$H(k) = \sum_{m=0}^{\frac{N}{2}-1} x(2m+1) W_{N/2}^{km}.$$

Then,

$$X(k) = G(k) + W_N^k H(k)$$

with $G(k)$ and $H(k)$ each being an $N/2-$point DFT which are periodic modulo $N/2$.

Now a direct DFT takes on the order of N^2 operations to compute. But, $G(k),\ k = 0, 1, 2, \ldots, N-1$ takes on the order of $(N/2)^2$ operations (remember $G(k)$ is periodic modulo $N/2$). $H(k)$ also takes on the order of $(N/2)^2$ operations. We need an additional N operations to add the two sums together. Thus, the total number of operations to computer $X(k)$ with this technique is

$$\text{Total} = (N/2)^2 + (N/2)^2 + N < N^2 \ \text{ for large N.}$$

If our number of samples is great enough, we can improve upon this by decomposing $G(k)$ and $H(k)$ each into two $N/4-$point DFTs. That is,

$$G(k) = \sum_{m=0}^{\frac{N}{4}-1} x(4m) W_{N/2}^{2km} + \sum_{m=0}^{\frac{N}{4}-1} x(4m+2) W_{N/2}^{k(2m+1)}$$

which may be written as

$$G(k) = \sum_{m=0}^{\frac{N}{4}-1} x(4m) W_{N/4}^{km} + W_{N/2}^{k} \sum_{m=0}^{\frac{N}{4}-1} x(4m+2) W_{N/4}^{km}.$$

Define

$$L(k) = \sum_{m=0}^{\frac{N}{4}-1} x(4m) W_{N/4}^{km}$$

and

$$M(k) = \sum_{m=0}^{\frac{N}{4}-1} x(4m+2) W_{N/4}^{km}$$

then

$$G(k) = L(k) + W_{N/2}^{k} M(k)$$

with $L(k)$ and $M(k)$ each being an $N/4-$point DFT which are periodic modulo $N/4$. We can repeat the above for $H(k)$.

We can then continue this process for as long as possible.

Computational Costs: Consider an $N-$point FFT where $N = 2^v$, v a positive integer. Then we have v stages in the Radix-2 DIT algorithm (which means we decimate by a factor of 2 in each stage). So the computional cost is on the order of $\frac{N}{2}v = \frac{N}{2}\log_2 N$ computations.