8.0 Sampling and Signal Reconstruction

Say we sample a signal $x_a(t)$, $t \in (-\infty, \infty)$, at sampling intervals T. We get samples $x_a(nT)$, $n = 0, \pm 1, \pm 2, \ldots$. Let $\{x(n)\} = \{x_a(nT)\}$, $n = 0, \pm 1, \pm 2, \ldots$.

$$x_a(t) \overset{\text{FT}}{\longleftrightarrow} X(\Omega)$$

$$x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\omega)$$

where,

$$X(\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-i\Omega t} dt, \quad x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{i\Omega t} d\Omega.$$

8.1 Relationship Between $X(\Omega)$ and $X(\omega)$

$$x_a(nT) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{i\Omega nT} d\Omega$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\Omega_0}^{\Omega_0} X(\Omega + 2k\Omega_0) e^{i(\Omega + 2k\Omega_0)nT} d\Omega.$$

Let $\Omega_0 = \pi/T$. Then, $e^{i2k\Omega_0 nT} = 1 \; \forall \; n$. Thus,

$$x_a(nT) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\Omega_0}^{\Omega_0} X(\Omega + 2k\Omega_0) e^{i\Omega nT} d\Omega.$$

Let $\omega = \Omega T$. Then, $\omega_0 = \Omega_0 T = \frac{\pi}{T} T = \pi$ and $d\omega = T d\Omega$. Get

$$x_a(nT) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega}{T} + \frac{2k\pi}{T}\right) e^{i\omega n} d\omega.$$

But,

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{i\omega n} d\omega.$$

Therefore, since $x(n) = x_a(nT)$, we see that

$$X(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega}{T} + \frac{2k\pi}{T}\right).$$

8.2 Sampling Theorem

From the above development we can deduce that $x_a(t)$ may be recovered from its samples, $x(n) = x_a(nT)$, if and only if

$$X(\Omega) = 0, \ |\Omega| \geq \pi/T.$$

If this is satisfied we have no aliasing and one period of $X(\omega)$ equals $\frac{1}{T}X(\omega/T)$.

8.3 Signal Reconstruction

8.3.1 Ideal Reconstruction

Here we seek to recover $x_a(t)$ from $x(n)$, assuming sampling at least Nyquist rate. So we assume

$$X(\Omega) = 0, \ |\Omega| \geq B \text{ (bandlimited)}.$$

Then,

$$x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{i\Omega t}d\Omega = \frac{1}{2\pi}\int_{-B}^{B} X(\Omega)e^{i\Omega t}d\Omega.$$

Using $\omega = \Omega T$, we may write

$$x_a(t) = \frac{1}{2\pi}\int_{-BT}^{BT} \frac{1}{T}X\left(\omega/T\right)e^{i\omega t/T}d\omega.$$

Now,

$$\frac{1}{T}X\left(\omega/T\right) = X(\omega) \quad \Big(\text{actually one period of } X(\omega)\Big)$$

and

$$X(\omega) = \sum_n x(n)e^{-i\omega n}.$$

Thus,

$$\begin{aligned} x_a(t) &= \frac{1}{2\pi}\int_{-BT}^{BT}\sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n}e^{i\omega t/T}d\omega \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} x(n)\int_{-BT}^{BT} e^{i\omega(t/T-n)}d\omega \end{aligned}$$

or

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n) \cdot \frac{\sin[BT(t/T - n)]}{\pi(t/T - n)}.$$

In the special case when $T = \pi/B$, this becomes

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n) \cdot \frac{\sin[\pi(t/T - n)]}{\pi(t/T - n)}.$$

This is a sinc interpolation formula.

8.3.2 Practical Reconstruction

Here we consider a sample and hold circuit (called a zeroth order hold).

Let

$$\hat{x}_a(t) = \frac{1}{T} x_a(nT) \quad \forall t \in [nT - T/2,\ nT + T/2]$$

and let $R(t)$ be the unit rectangle defined as

$$R(t) = \begin{cases} 1, & |t| \leq T/2 \\ 0, & \text{elsewhere.} \end{cases}$$

Then,

$$\hat{x}_a(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} x_a(nT) R(t - nT)$$

and

$$\hat{X}(\Omega) = \frac{1}{T} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_a(nT) R(t - nT) e^{-i\Omega(t-nT)} e^{-i\Omega nT} dt.$$

Now

$$\frac{1}{T} \int_{-\infty}^{\infty} R(t - nT) e^{-i\Omega(t-nT)} dt = \frac{1}{T} \int_{-T/2}^{T/2} e^{-i\Omega t} dt = \frac{\sin[\Omega T/2]}{\Omega T/2}.$$

Therefore,

$$\hat{X}(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-i\Omega nT} \cdot \frac{\sin[\Omega T/2]}{\Omega T/2}$$

or

$$\hat{X}(\Omega) = X(\omega)\Big|_{\omega=\Omega T} \cdot \frac{\sin[\Omega T/2]}{\Omega T/2}.$$