7.0 Discrete Fourier Transform

7.1 Definitions

Definition: Given a finite sequence $x(n),\ n = 0, 1, \ldots,\ N-1$, let $W_N = e^{-i2\pi/N}$. Then,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad 0 \le k \le N-1$$

and

$$x(n) = \frac{1}{N} \sum_{n=0}^{N-1} X(k) W_N^{-nk}, \quad 0 \le n \le N-1$$

constitute a discrete Fourier transform pair.

Matrix Form Construct the $N \times N$ matrix $\mathbf{W}_N = \left[W_N^{nk} \right]$.

Let

$$\mathbf{x}_N = [x(0),\ x(1),\ \ldots,\ x(N-1)]^T$$

and

$$\mathbf{X}_N = [X(0),\ X(1),\ \ldots,\ X(N-1)]^T$$

Then,

$$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N$$

and

$$\mathbf{x}_N = \mathbf{W}_N^{-1} \mathbf{X}_N.$$

Notes

i. $\mathbf{W}_N^{-1} = \dfrac{1}{N} \mathbf{W}_N^{*^T} = \dfrac{1}{N} \mathbf{W}_N^{*}, \quad \left(\mathbf{W}_N \mathbf{W}_N^{*} = N \mathbf{I}_N \right).$

ii. The $(n,k)^{th}$ element of $\mathbf{W}_N$ is W_N^{nk} and the $(n,k)^{th}$ element of $\mathbf{W}_N^{-1}$ is W_N^{-nk}.

iii. The book says $\mathbf{W}_N$ is an orthogonal (unitary) matrix. But $\mathbf{U}$ unitary usually means $\mathbf{U} \cdot \mathbf{U}^{*^T} = \mathbf{I}_N$ (a real matrix $\mathbf{V}$ is orthogonal if $\mathbf{V} \cdot \mathbf{V}^T = \mathbf{I}_N$). So, $\mathbf{W}_N$ is not unitary by this definition (but, $\frac{1}{\sqrt{N}} \mathbf{W}_N$ is unitary).

Claim: Let

$$\underline{W}_N^k = \left[W_N^0,\ W_N^k,\ W_N^{2k}, \ldots,\ W_N^{(N-1)k}\right]^T.$$

Then, $\underline{W}_N^k,\ k = 0, 1, \ldots, N-1$ are a set of basis vectors. In fact,

$$\underline{W}_N^{k*^T} \underline{W}_N^m = N\delta[(k-m) \text{mod } N].$$

Proof: Let $(k-m) = l + pN,\ 0 \leq l \leq N-1$. Then,

$$\underline{W}_N^{k*^T} \underline{W}_N^m = \sum_{n=0}^{N-1} e^{\frac{i2\pi kn}{N}} e^{-\frac{i2\pi mn}{N}} = \sum_{n=0}^{N-1} e^{\frac{i2\pi(l+pN)n}{N}} = \sum_{n=0}^{N-1} e^{\frac{i2\pi ln}{N}} = N\delta(l).$$

So, $(k-m) = pN \Rightarrow k \equiv m \text{ mod } N \Rightarrow \underline{W}_N^{k*^T} \underline{W}_N^m = N\delta[(k-m) \text{mod } N]$.

Thus, the $\underline{W}_N^k$ are a complete basis for the space $\mathbf{C}^N$ of $N-$dim complex vectors, i.e., any vector $\mathbf{X}_N \in \mathbf{C}^N$ can be written

$$\mathbf{X}_N = \sum_{n=0}^{N-1} \alpha_n \underline{W}_N^k.$$

But this is the DFT, i.e., $\alpha_n = x(n)$.

So, if we define

$$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N$$

then

$$\mathbf{x}_N = \mathbf{W}_N^{-1} \mathbf{X}_N = \frac{1}{N} \mathbf{W}_N^{*^T} \mathbf{X}_N = \frac{1}{N} \mathbf{W}_N^* \mathbf{X}_N.$$

There are other discrete transforms we could use for signal analysis but the DFT is desirable because

1. it gives the spectral content of the signal
2. of its relation to other Fourier transforms
3. there exist a fast computation (FFT).

7.2 Properties of the DFT

Periodicity: For any integer m,

$$X(k+mN) = X(k),$$

$$x(n+mN) = x(n).$$

To see the first part we compute

$$X(k+mN) = \sum_{n=0}^{N-1} x(n)e^{-\frac{i2\pi n(k+mN)}{N}} = \sum_{n=0}^{N-1} x(n)e^{-\frac{i2\pi nk}{N}} e^{-i2\pi mn}$$

$$= \sum_{n=0}^{N-1} x(n)e^{-\frac{i2\pi nk}{N}} = X(k).$$

Linearity:

$$ax_1(n) + bx_2(n) \overset{DFT}{\longleftrightarrow} aX_1(k) + bX_2(k).$$

Shift:

$$x[(n-m)\text{mod } N] \overset{DFT}{\longleftrightarrow} e^{-i2\pi mk/N} X(k).$$

See the text for a proof.

Circular Convolution:

Definition: The *circular convolution of* $x(n)$ *and* $y(n)$ is

$$x(n) \circledast y(n) = \sum_{m=0}^{N-1} x(m)y[(n-m)\text{mod } N].$$

Let $z(n) = x(n) \circledast y(n)$. Then,

$$z(n+lN) = \sum_{m=0}^{N-1} x(m)y[(n+lN-m)\text{mod } N]$$

$$= \sum_{m=0}^{N-1} x(m)y[(n-m)\text{mod } N] = z(n).$$

Thus, $x(n) \circledast y(n)$ is N-periodic.

Let us look at the DFT of $z(n)$. We get

$$Z(k) = \sum_{n=0}^{N-1} z(n)e^{-i2\pi nk/N} = \sum_{n=0}^{N-1} z(n)W_N^{kn}$$

$$= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} x(m)y[(n-m)\text{mod } N]W_N^{kn}$$

$$= \sum_{m=0}^{N-1} x(m) \sum_{n=0}^{N-1} y[(n-m)\text{mod } N]W_N^{kn}.$$

But,

$$y[(n-m)\text{mod } N] \overset{DFT}{\longleftrightarrow} e^{-i2\pi mk/N}Y(k) \text{ by shift property.}$$

So,

$$Z(k) = \sum_{m=0}^{N-1} x(m)e^{-i2\pi mk/N}Y(k)$$

or

$$Z(k) = X(k)Y(k).$$

Therefore,

$$x(n) \circledast y(n) \overset{DFT}{\longleftrightarrow} X(k)Y(k).$$

7.3 Relationship Between DFT and DTFT

7.3.1 Finite Sequences

Say $x_N(0) = 0$ for $n < 0,\ n \geq N$. Then,

$$X_N(k) = \sum_{n=0}^{N-1} x_N(n)e^{-i2\pi nk/N} = X_N(\omega)\Big|_{\omega=2\pi k/N}.$$

So, the DFT of $x_N(n)$ gives samples of the DTFT of $x_N(n)$. Now,

$$X_N(\omega) = \sum_{n=0}^{N-1} x_N(n)e^{-i\omega n} = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X_N(k)e^{i2\pi nk/N}\right]e^{-i\omega n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X_N(k) \sum_{n=0}^{N-1} e^{i2\pi nk/N} e^{-i\omega n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X_N(k) \frac{1 - e^{i(2\pi k/N - \omega)N}}{1 - e^{i(2\pi k/N - \omega)}}$$

or,

$$X_N(\omega) = \frac{1}{N} \sum_{k=0}^{N-1} X_N(k) \frac{1 - e^{-i\omega N}}{1 - e^{-i(\omega - 2\pi k/N)}}.$$

This last expression shows how we can compute the DTFT from the DFT for finite length sequences.

7.3.2 Infinite Sequences

Given an infinite sequence, $x(n)$, we can compute the DTFT (if it exists), but to compute the DFT we can only work with finite length sequences. So, if we do have an infinite sequence we must truncate it which means we cannot recover the DTFT from the DFT in general.

Let

$$x_N(n) = \begin{cases} x(n), & 0 \le n \le N-1 \\ 0, & \text{elsewhere.} \end{cases}$$

We may write this as

$$x_N(n) = x(n)w(n)$$

where,

$$w(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{elsewhere.} \end{cases}$$

Then,

$$X_N(\omega) = X(\omega) * W(\omega)$$

where,

$$W(\omega) = \sum_{n=0}^{N-1} e^{-i\omega n} = \frac{1 - e^{-i\omega N}}{1 - e^{-i\omega}} = \frac{e^{-i\omega N/2}}{e^{-i\omega/2}} \frac{e^{i\omega N/2} - e^{-i\omega N/2}}{e^{i\omega/2} - e^{-i\omega/2}}$$

$$= \frac{\sin(\omega N/2)}{\sin(\omega/2)} e^{-i\omega(N-1)/2}.$$

We can then compute the DFT for this truncated sequence as

$$X_N(k) = X_N(\omega)\Big|_{\omega=2\pi k/N}.$$

Example: Let

$$x(n) = e^{i\omega_0 n}.$$

Find $X_N(k)$.

Solution: We know

$$X_N(\omega) = X(\omega) * W(\omega).$$

Now

$$X(\omega) = \sum_{n=-\infty}^{\infty} e^{i\omega_0 n} e^{-i\omega n} = \sum_{n=-\infty}^{\infty} e^{-i(\omega-\omega_0)n}.$$

To evaluate this last sum we need the following result.

Claim:

$$\sum_{n=-\infty}^{\infty} e^{-i(\omega-\omega_0)n} = 2\pi \sum_{n=-\infty}^{\infty} \delta\left(\omega - \omega_0 - 2n\pi\right).$$

Proof: We know that the inverse DTFT of the LHS of the above is $x(n) = e^{-\omega_0 n}$. So to prove the claim we will show that the inverse DTFT of the RHS is also this same sequence.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi \sum_{n=-\infty}^{\infty} \delta\left(\omega - \omega_0 - 2n\pi\right) e^{i\omega m} d\omega$$

$$= \int_{-\pi}^{\pi} \delta\left(\omega - \omega_0\right) e^{i\omega m} d\omega$$

$$= e^{i\omega_0 m} = x(m)$$

which is the desired result.

Therefore,

$$X_N(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(\omega'\right) W\left(\omega - \omega'\right) d\omega'$$

$$= \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} \delta\left(\omega' - \omega_0 - 2n\pi\right) \frac{\sin\left[(\omega-\omega')N/2\right]}{\sin\left[(\omega-\omega')/2\right]} e^{-i(\omega-\omega')(N-1)/2} d\omega'$$

$$= \int_{-\pi}^{\pi} \delta\left(\omega' - \omega_0\right) \frac{\sin\left[(\omega - \omega')N/2\right]}{\sin\left[(\omega - \omega')/2\right]} e^{-i(\omega-\omega')(N-1)/2} d\omega'$$

$$= \frac{\sin\left[(\omega - \omega_0)N/2\right]}{\sin\left[(\omega - \omega_0)/2\right]} e^{-i(\omega-\omega_0)(N-1)/2}$$

and

$$X_N(k) = X_N(\omega)\Big|_{\omega = 2\pi k/N}.$$

Note that

$$\Big|X_N(\omega)\Big| = \frac{\sin\left[(\omega - \omega_0)N/2\right]}{\sin\left[(\omega - \omega_0)/2\right]}.$$

Thus,

$$\Big|X_N(k)\Big| = \frac{\sin\left[(2\pi k/N - \omega_0)N/2\right]}{\sin\left[(2\pi k/N - \omega_0)/2\right]}.$$

Observe that

$$\text{if } \omega_0 = \frac{2\pi l}{N},\ l \text{ an integer, then}$$

$$X_N(k) = \begin{cases} 0, & k \neq l \bmod N \\ N, & k = l \bmod N. \end{cases}$$

7.4 Zero Padding

In cases like that just mentioned we would like to have more samples of the DTFT spectrum in order to better "see" the continuous spectrum. We can do this by zero padding the finite length sequence, $x_N(n)$.

Let

$$x_{2N}(n) = \begin{cases} x_N(n), & 0 \leq n \leq N-1 \\ 0, & N \leq n \leq 2N-1 \end{cases}$$

where, $x_N(n) = x(n)w(n)$ with $w(n)$ being the window function as defined above. Then,

$$X_{2N}(k) = \sum_{n=0}^{N-1} x_{2N}(n) e^{-i2\pi nk/2N} = X_N(\omega)\Big|_{\omega=2\pi k/2N}.$$

Note that zero padding improves the readibility of results but does *not* add any new information.