6.0 Frequency Analysis

6.1 Continuous-time Systems

The Fourier transform of $x(t)$ is

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt$$

and

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{i2\pi f t} df$$

is the inverse, provided these exist. The Fourier transform gives a measure of the sinusoidal components making up the Fourier series of the periodically replicated version of $x(t)$.

6.2 Discrete-time Systems

6.2.1 Fourier Transform

Definition: The *discrete-time Fourier transform* (DTFT) of a sequence $x(n)$ is

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-i\omega n}.$$

$X(\omega)$ gives the frequency content of $x(n)$.

Note: If the ROC of the z-transform includes the unit circle then

$$X(\omega) = X(z)\Big|_{z=e^{i\omega}}.$$

Another notation for the DTFT is $X\left(e^{i\omega}\right)$.

Example: Let $x(n) = \alpha^n u(n)$. Then,

$$X(\omega) = \sum_{n=0}^{\infty} \alpha^n e^{-i\omega n} = \sum_{n=0}^{\infty} \left(\alpha e^{-i\omega}\right)^n = \frac{1}{1 - \alpha e^{-i\omega}}, \quad |\alpha| < 1,$$

and

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad |\alpha| < |z|.$$

So, if $|\alpha| < 1$ then $X(\omega) = X(z)\Big|_{z=e^{i\omega}}$.

Remarks:

1. $e^{-i\omega n} = e^{-i(\omega+2\pi)n}$ since $e^{-i2\pi n} = 1\ \forall\ n \in \mathbf{Z}$. So, the DTFT is periodic of period 2π.

2. $X(\omega)$ is complex, i.e.,

$$X(\omega) = X_R(\omega) + iX_I(\omega).$$

   Thus,

$$|X(\omega)| = \sqrt{X_R^2(\omega) + X_I^2(\omega)},$$

$$|X(\omega)|^2 = X(\omega)X^*(\omega), \quad \angle X(\omega) = \tan^{-1}\left(\frac{X_I(\omega)}{X_R(\omega)}\right).$$

Example: Consider

$$X(\omega) = \frac{1}{1 - \alpha e^{-i\omega}}, \quad |\alpha| < 1.$$

Now,

$$\frac{1}{1-\alpha e^{-i\omega}} \frac{1-\alpha e^{i\omega}}{1-\alpha e^{i\omega}} = \frac{1-\alpha e^{i\omega}}{1-2\alpha\cos\omega+\alpha^2} = \frac{1-\alpha\cos\omega - i\alpha\sin\omega}{1-2\alpha\cos\omega+\alpha^2}.$$

So,

$$X_R(\omega) = \frac{1-\alpha\cos\omega}{1-2\alpha\cos\omega+\alpha^2}, \quad X_I(\omega) = \frac{-\alpha\sin\omega}{1-2\alpha\cos\omega+\alpha^2}.$$

Also,

$$|X(\omega)|^2 = \frac{1}{1-2\alpha\cos\omega+\alpha^2}, \quad \angle X(\omega) = -\tan^{-1}\left(\frac{\alpha\sin\omega}{1-\alpha\cos\omega}\right).$$

If the DTFT is a rational function of $e^{-i\omega}$, we can take the inverse transform by partial fractions as with the inverse z-transform. More generally, we proceed as follows.

$$X(\omega) = \sum_{k=-\infty}^{\infty} x(k)e^{-i\omega k}.$$

So,

$$\int_{-\pi}^{\pi} X(\omega)e^{i\omega n}d\omega = \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} x(k)e^{-i\omega k}e^{i\omega n}d\omega$$
$$= \sum_k x(k) \int_{-\pi}^{\pi} e^{i\omega(n-k)}d\omega$$
$$= \sum_k x(k)2\pi\delta(n-k)$$
$$= x(n)2\pi.$$

This then justifies the following definition.

Definition: The *inverse discrete-time Fourier transform* (inverse DTFT) of $X(\omega)$ is

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{i\omega n}d\omega.$$

The above development shows that $x(n) \longleftrightarrow X(\omega)$ form a discrete-time Fourier transform pair with the given defintions.

Example: Let

$$X(\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \text{elsewhere.} \end{cases}$$

Find $x(n)$.

Solution.

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{i\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{i\omega n}d\omega = \frac{1}{2\pi i n}e^{i\omega n}\Big|_{-\omega_c}^{\omega_c}$$
$$= \frac{1}{2\pi i n}\left[e^{i\omega_c n} - e^{-i\omega_c n}\right] = \frac{\sin\omega_c n}{\pi n}, \quad n \ne 0.$$

For $n = 0$,

$$x(0) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi}.$$

So,

$$x(n) = \begin{cases} \dfrac{\omega_c}{\pi}, & n = 0 \\ \dfrac{\sin\omega_c n}{\pi n}, & n \ne 0. \end{cases}$$

Now given the $x(n)$ just derived let us try to calculate $X(\omega)$.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\sin \omega_c n}{\pi n} e^{-i\omega n} + \frac{\omega_c}{\pi}.$$

This last expression does not converge uniformly for all ω but does converge in the mean-square sense to

$$X(\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \text{elsewhere.} \end{cases}$$

Let

$$X_N(\omega) = \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{\sin \omega_c n}{\pi n} e^{-i\omega n} + \frac{\omega_c}{\pi}.$$

Then,

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} |X(\omega) - X_N(\omega)|^2 \, d\omega = 0.$$

$X_N(\omega)$ is the DTFT of $x_N(n)$, where

$$x_N(n) = \begin{cases} x(n), & -N \leq n \leq N \\ 0, & \text{elsewhere.} \end{cases}$$

Define the window function

$$w(n) = \begin{cases} 1, & -N \leq n \leq N \\ 0, & \text{elsewhere.} \end{cases}$$

Then, $x_N(n) = x(n)w(n)$ and $X_N(\omega) = X(\omega) * W(\omega)$.

Now,

$$W(\omega) = \sum_{n=-N}^{N} e^{-i\omega n} = \frac{e^{i\omega N} - e^{-i\omega(N+1)}}{1 - e^{-i\omega}}$$

$$= \frac{e^{-i\omega/2}}{e^{-i\omega/2}} \cdot \frac{e^{i\omega/2}e^{i\omega N} - e^{-i\omega N}e^{-i\omega/2}}{e^{i\omega/2} - e^{-i\omega/2}}$$

$$= \frac{e^{i\omega(N+1/2)} - e^{-i\omega(N+1/2)}}{e^{i\omega/2} - e^{-i\omega/2}}$$

which becomes

$$W(\omega) = \frac{\sin[\omega(N+1/2)]}{sin(\omega/2)}.$$

So,

$$X_N(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\theta)W(\omega-\theta)d\theta$$

or

$$X_N(\omega) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} \frac{\sin[(\omega-\theta)(N+1/2)]}{sin[(\omega-\theta)/2]}d\theta.$$

A plot of this function shows an overshoot at $\omega = \pm\omega_c$.

Sometimes it is desirable to know the energy in a sequence. The energy may be computed from the sequence itself or from its DTFT. This follows from Parseval's theorem.

Parseval's Theorem

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega.$$

Proof:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{n=-\infty}^{\infty} x_1(n)e^{-i\omega n}\right]X_2^*(\omega)d\omega$$

$$= \sum_{n=-\infty}^{\infty} x_1(n)\frac{1}{2\pi}\int_{-\pi}^{\pi} X_2^*(\omega)e^{-i\omega n}d\omega = \sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n).$$

Special case: Let $x(n) = x_1(n) = x_2(n)$. Then,

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega.$$

6.2.2 Frequency Response

Definition: The *frequency response* of a stable LTI system is the DTFT of its impulse response sequence, i.e.,

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n}.$$

Now

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) x(n-k).$$

Let $x(n) = e^{i\omega_0 n}$. Then,

$$y(n) = \sum_k h(k) e^{i\omega_0 (n-k)} = e^{i\omega_0 n} \sum_k h(k) e^{-i\omega_0 k} = e^{i\omega_0 n} H(\omega_0).$$

Note: Exponential sequences are called *eigenfunctions* of LTI systems.

Now

$$H(\omega_0) = |H(\omega_0)| e^{i\angle H(\omega_0)}.$$

So,

$$e^{i\omega_0 n} H(\omega_0) = |H(\omega_0)| e^{i\left[\omega_0 n + \angle H(\omega_0)\right]}.$$

Also, if $x(n) = e^{-i\omega_0 n}$, then

$$y(n) = e^{-i\omega_0 n} H(-\omega_0)$$

where,

$$H(-\omega_0) = |H(\omega_0)| e^{-i\angle H(\omega_0)}.$$

Since

$$\cos \omega_0 n = \frac{e^{i\omega_0 n} - e^{-i\omega_0 n}}{2}$$

it follows that if $x(n) = \cos \omega_0 n$, then

$$y(n) = |H(\omega_0)| \cos\left[\omega_0 n + \angle H(\omega_0)\right].$$

Example: Let $h(n) = \{1, 0, 1\}$. Then,

$$H(\omega) = \left(e^{i\omega} + e^{-i\omega}\right) = 2\cos\omega.$$

Suppose $x(n) = \cos\left(\frac{\pi}{3}n\right)$. Then,

$$|H(\pi/3)| = |2\cos(\pi/3)| = 1$$

and

$$y(n) = \cos\left(\frac{\pi}{3}n\right).$$

6.2.3 Geometric Interpretation of Frequency Response

Consider

$$H(z) = A \cdot \frac{\prod_{k=1}^{M}\left(1 - c_k z^{-1}\right)}{\prod_{j=1}^{N}\left(1 - d_j z^{-1}\right)}.$$

If the unit circle is in the ROC, then

$$H(\omega) = H(z)\Big|_{z=e^{i\omega}}.$$

In this case on the unit circle

$$\left|1 - c_k z^{-1}\right| = \left|1 - c_k e^{-i\omega}\right| = \left|e^{i\omega}\right| \cdot \left|1 - c_k e^{-i\omega}\right| = \left|e^{i\omega} - c_k\right|.$$

So,

$$|H(\omega)| = |A| \cdot \frac{\prod_{k=1}^{M}\left|e^{i\omega} - c_k\right|}{\prod_{j=1}^{N}\left|e^{i\omega} - d_j\right|}.$$

We can interpret this expression as

$$|H(\omega)| = |A| \cdot \frac{\prod_{k=1}^{M}\left(\text{distance from } k^{th} \text{ zero to unit circle point } e^{i\omega}\right)}{\prod_{j=1}^{N}\left(\text{distance from } j^{th} \text{ pole to unit circle point } e^{i\omega}\right)}$$

and the phase response is

$$\angle H(\omega) = \sum_{k=1}^{M} \arg\left(1 - c_k e^{-i\omega}\right) - \sum_{j=1}^{N} \arg\left(1 - d_j e^{-i\omega}\right).$$

We can now make the following observations:

1. The poles of a low pass filter should be located close to $\omega = 0$ while the zeros should be located close to $\omega = \pm\pi$.
2. The poles of a high pass filter should be located close to $\omega = \pm\pi$ while the zeros should be located close to $\omega = 0$.

Example: Let

$$H(z) = \frac{1 + 0.8z^{-1}}{1 - 0.9z^{-1}}, \quad \text{ROC} = \{z : |z| > 0.9\}.$$

Then,

$$\begin{aligned} H(\omega) &= \frac{1 + 0.8e^{-i\omega}}{1 - 0.9e^{-i\omega}} \\ \left|1 + 0.8e^{-i\omega}\right| &= \left|e^{i\omega} + 0.8\right|, \\ \left|1 - 0.9e^{-i\omega}\right| &= \left|e^{i\omega} - 0.9\right|. \end{aligned}$$

We have a zero at $z = -0.8$ and a pole at $z = 0.9$. We also have a zero and pole at $z = 0$ which cancel. A plot of $|H(\omega)|$ shows this is a LPF.

Definition: The *group delay* is defined as

$$\tau(\omega) = -\frac{d\angle H(\omega)}{d\omega}.$$

Note: A system with linear phase has a constant group delay. $\tau(\omega)$ represents the time delay that a signal component of frequency ω experiences as it passes thru the system. We usually would like for all frequencies to undergo the same delay so linear phase is often an important parameter in filter design.

Ideal LPF Let

$$H(\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \text{elsewhere.} \end{cases}$$

This implies

$$h(n) = \begin{cases} \dfrac{\omega_c}{\pi}, & n = 0 \\ \\ \dfrac{\sin \omega_c n}{\pi n}, & n \neq 0. \end{cases}$$

Define

$$\text{sinc}\phi = \begin{cases} 1, & n = 0 \\ \dfrac{\sin \phi}{\phi}, & n \neq 0. \end{cases}$$

Then,

$$h(n) = \frac{\omega_c}{\pi}\text{sinc}\left(\omega_c n\right).$$

More generally, we allow for constant gain and linear phase response and thus define

$$H_{LP}(\omega) = \begin{cases} Ae^{-i\omega k_0}, & |\omega| \leq \omega_c \\ 0, & \text{elsewhere.} \end{cases}$$

Then,

$$h_{LP}(n) = A\frac{\omega_c}{\pi}\text{sinc}\left(\omega_c\left(n - k_0\right)\right).$$

Note that $h_{LP}(n)$ is non-causal and so is not physically realizable. However, by allowing for smoother transitions in the frequency domain we can still design good low pass filters which are physically realizable.

Definition: An *all-pass system* is one which has a unity magnitude response over the entire frequency range $[-\pi,\ \pi]$.

Example: $H(z) = 1$ and $H(z) = z^{-k}$ represent all-pass systems since

$$\left| H(z)|_{z=e^{i\omega}} \right| = 1.$$

More generally, consider

$$H(z)=\frac{\sum_{k=0}^{N} a_k z^{-N+k}}{\sum_{k=0}^{N} a_k z^{-k}} = z^{-N}\frac{D\left(z^{-1}\right)}{D(z)}$$

where,

$$D(z)=\sum_{k=0}^{N} a_k z^{-k}.$$

Now

$$\left|H(\omega)\right| = \left|H(z)|_{z=e^{i\omega}}\right| = \left|e^{-i\omega N}\frac{D\left(e^{-i\omega}\right)}{D\left(e^{i\omega}\right)}\right|.$$

If the a_k coefficients are real, then $D\left(e^{i\omega}\right)=D^*\left(e^{-i\omega}\right)$ and thus $\left|H(\omega)\right|=1$ which is an all-pass system.

Let us now factor $D(z)$ as (assume N even)

$$D(z)=\prod_{i=1}^{N/2}\left(\alpha_{2i}z^{-2}+\alpha_{1i}z^{-1}+1\right).$$

If N is not even then use $\frac{N-1}{2}$ for $\frac{N}{2}$ and include the degree 1 factor.

For real coefficients we have complex conjugate poles $d_i,\ d_i^*$. So,

$$D(z)=\prod_{i=1}^{N/2}\left(1-d_i z^{-1}\right)\left(1-d_i^* z^{-1}\right)$$

$$\Rightarrow \alpha_{2i}=\left|d_i\right|^2,\quad \alpha_{1i}=-2\mathrm{Re}\left\{d_i\right\}.$$

Now, if $D(z)=0$ at $z=d_i$, then $D(1/z)=0$ at $z=1/d_i$.

Result: In all-pass systems, poles and zeros occur in reciprocal pairs.

Definition: A system with all poles and zeros inside the unit circle is called a *minimum phase system.*

Claim: Any LTI system function $H(z)$ can be written as

$$H(z) = H_{MP}(z)H_{AP}(z)H_{UC}(z)$$

where, $H_{MP}(z)$ is minimum phase, $H_{AP}(z)$ is all-pass, $H_{UC}(z)$ has all poles and zeros on the unit circle.

***Proof*:** See text.

Except for a possible scaling factor that occurs when we have poles/zeros on the unit circle, $H(z)$ and $H_{MP}(z)$ have the same magnitude response. Among all systems with the same magnitude response (up to a scaling factor) the minimum phase system is the one with the smallest group delay since

$$\tau(\omega) = \tau_{MP}(\omega) + \tau_{AP}(\omega) + \tau_{UC}(\omega)$$

and both $\tau_{AP}(\omega)$ and $\tau_{UC}(\omega)$ are each greater than or equal to zero (see pg. 352 of text).