5.0 Z-transform

5.1 Introduction

The z-transform is another tool that aids us in signal analysis and filter design. The z-transform exists for a broader class of signals than the discrete-time Fourier transform (DTFT) which will be studied later. We will also see a relationship between the z-transform and the DTFT.

5.2 Forward Z-transform

Definition: The z-transform, $X(z)$, of a sequence, $x(n)$, is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

whenever this sum is bounded. This definition also requires that the values of the complex number z be specified for which the sum exists (if any such z exists at all).

Definition: The set of all z for which the above sum exists is called the *region of convergence* (ROC).

Special Case: If $x(n)$ is a finite sequence, say $x(n)$ is defined for $N_1 \leq n \leq N_2,\ N_1,\ N_2 \in \mathbf{Z}$, then

$$X(z) = \sum_{n=N_1}^{N_2} x(n) z^{-n}$$

and the ROC is all z except for possibly $z = \infty$ and/or $z = 0$.

Example: Let

$$x(n) = \alpha^n u(n).$$

Then,

$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^n u(n) z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} \left(\alpha z^{-1}\right)^n$$

$$= \frac{1}{1 - \alpha z^{-1}} = \frac{z}{z - \alpha}$$

provided

$$\left|\alpha z^{-1}\right| < 1 \Rightarrow \text{ROC} = \{z : |z| > |\alpha|\}.$$

$x(n)$ is an example of a right-sided sequence.

<u>Definition:</u> $x(n)$ is called a *right-sided sequence* if $\exists\ n_0$ such that $x(n) = 0\ \forall\ n < n_0$.

Here the ROC is the exterior of a disc.

Example: Let

$$x(n) = -\alpha^n u(-n-1).$$

Then,

$$X(z) = \sum_{n=-\infty}^{\infty} -\alpha^n u(-n-1) z^{-n} = -\sum_{n=-\infty}^{-1} \alpha^n z^{-n} = -\sum_{n=1}^{\infty} \left(\alpha^{-1} z\right)^n$$

$$= -\frac{\alpha^{-1} z}{1-\alpha^{-1} z} = \frac{z}{z-\alpha}$$

provided

$$\left|\alpha^{-1} z\right| < 1 \Rightarrow \text{ROC} = \{z : |z| < |\alpha|\}.$$

This $x(n)$ is an example of a left-sided sequence.

<u>Definition:</u> $x(n)$ is called a *left-sided sequence* if $\exists\ n_0$ such that $x(n) = 0\ \forall\ n > n_0$.

Here the ROC is the interior of a disc.

Note: Let $\mathcal{Z}$ denote the z-transform operator. Then, from the last two examples we see that

$$\mathcal{Z}\left[\alpha^n u(n)\right] = \mathcal{Z}\left[-\alpha^n u(-n-1)\right]$$

except for their ROC. So we must always specify the ROC of the z-transform so that we can uniquely associate it with the sequence from which it came.

Initial Value Theorem: If $x(n) = 0$ for $n < 0$ (i.e., $x(n)$ is a causal sequence) then

$$x(0) = \lim_{z \to \infty} X(z).$$

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

so

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \Big(x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots \Big) = x(0).$$

5.3 Properties of the Z-transform

Linearity:

$$\mathcal{Z}\left[ax_1(n) + bx_2(n)\right] = a\mathcal{Z}\left[x_1(n)\right] + b\mathcal{Z}\left[x_2(n)\right].$$

Shift:

$$x\left(n - n_0\right) \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0} X(z).$$

Proof:

$$\mathcal{Z}\left[x\left(n-n_0\right)\right] = \sum_{n=-\infty}^{\infty} x\left(n-n_0\right) z^{-n} \quad [\text{let } n' = n - n_0]$$

$$= \sum_{n'=-\infty}^{\infty} x\left(n'\right) z^{-(n'+n_0)} = z^{-n_0} X(z).$$

$$ROC = ROC_x \text{ except possibly at } z = 0 \text{ or } z = \infty.$$

Convolution:

$$x_1(n) * x_2(n) \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z) X_2(z).$$

Proof:

$$\sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k) z^{-n} = \sum_{k=-\infty}^{\infty} x_1(k) \left[\sum_{n=-\infty}^{\infty} x_2(n-k) z^{-n} \right]$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) z^{-k} X_2(z) = X_1(z) X_2(z).$$

Dual:

$$x_1(n)x_2(n) \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z) * X_2(z).$$

Derivative:

$$nx(n) \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{d}{dz}X(z).$$

Proof:

$$\sum_{n=-\infty}^{\infty} nx(n)z^{-n} = z\sum_{n=-\infty}^{\infty} nx(n)z^{-(n+1)}.$$

Now,

$$X(z) = \sum_n x(n)z^{-n} \Rightarrow \frac{d}{dz}X(z) = -\sum_n nx(n)z^{-(n+1)}$$

so the result follows.

$$ROC = ROC_x \text{ except possibly at } z = 0 \text{ or } z = \infty.$$

Scaling:

$$a^n x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z/a).$$

Proof:

$$\sum_n a^n x(n)z^{-n} = \sum_n x(n)(z/a)^{-n} = X(z/a).$$

If initial ROC was $r_1 < |z| < r_2$ then new ROC is $r_1 < |z/a| < r_2$.

Symmetry: Let $x(n)$ be real (so $x(n) = x^*(n)$).

$$X(z) = \sum_n x(n)z^{-n}$$

so

$$X^*(z) = \sum_n x^*(n){z^*}^{-n} = \sum_n x(n){z^*}^{-n} \Rightarrow X^*(z) = X(z^*).$$

Example: Evaluate the following infinite sum using z-transform properties:

$$S = \sum_{n=0}^{\infty} n^2(1/2)^n.$$

Solution: Let us write S as

$$S = \sum_{n=0}^{\infty} n^2 2^{-n}.$$

Let $x(n) = u(n)$ and let $x_1(n) = nx(n)$. Then,

$$X_1(z) = -z\frac{d}{dz}X(z).$$

Let $x_2(n) = n^2 x(n) = nx_1(n)$. Then,

$$X_2(z) = -z\frac{d}{dz}X_1(z) = -z\frac{d}{dz}\left[-z\frac{d}{dz}X(z)\right].$$

Now,

$$X(z) = \mathcal{Z}\left[u(n)\right] = \frac{1}{1-z^{-1}},\ |z| > 1,$$

$$\Rightarrow X_1(z) = \frac{z^{-1}}{\left(1-z^{-1}\right)^2} \Rightarrow X_2(z) = \frac{z^{-1}+z^{-2}}{\left(1-z^{-1}\right)^3}.$$

We note that

$$S = X_2(z)\Big|_{z=2} \Rightarrow S = 6.$$

5.4 Some Results From Complex Variable Theory

To take the inverse z-transform directly we need to use complex analysis.

Definition: A function of the complex variable z is *analytic* at a point z_0 if its derivative exists at z_0 and there exists some neighborhood of z_0 in all of whose points f is also differentiable.

Example: $f(z) = z^2$ is analytic everywhere.

Example: $f(z) = |z|^2$ is analytic nowhere. Why? Consider $f'(z)$.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \text{ if this limit exists.}$$

Let

$$h(z) = \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$
$$= \frac{(z+\Delta z)\left(z^* + (\Delta z)^*\right) - zz^*}{\Delta z}$$
$$= z^* + (\Delta z)^* + z\frac{(\Delta z)^*}{\Delta z}.$$

If the above limit exists as $\Delta z \to 0$, we can let $\Delta z = \Delta x + i\Delta y$ approach 0 in any manner.

i. Let $\Delta z = \Delta x + i0$ (approach 0 along real axis). So, $(\Delta z)^* = \Delta z$. We get $h_1(z) = z^* + z$.

ii. Let $\Delta z = 0 + i\Delta y$ (approach 0 along imaginary axis). Thus, $(\Delta z)^* = -\Delta z$. We get $h_2(z) = z^* - z$.

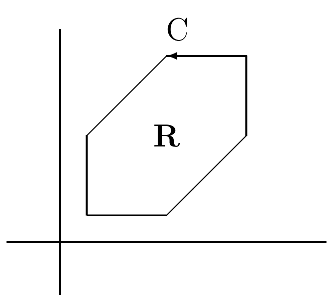
The limit must be unique. Therefore,

$$h_1(z) = h_2(z) \Rightarrow z^* + z = z^* - z \Rightarrow z = 0.$$

So, $f'(z)$ exists only at the origin and $f'(z)$ does not exist in any other point in a neighborhood of the origin $\Rightarrow f(z) = |z|^2$ is analytic nowhere.

Cauchy-Goursat Theorem: If a function f is analytic in a region R and on its boundary C, then

$$\oint_C f(z)dz = 0.$$



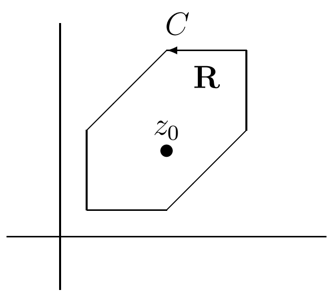
Convention: The positive direction of transversing a path is the counter clockwise direction and will be denoted with a down arrow $\downarrow$. The clockwise direction will be denoted with an up arrow $\uparrow$.

Cauchy Integral Formula

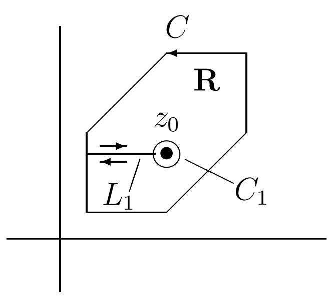
Consider

$$g(z) = \frac{f(z)}{z - z_0}$$

where $f(z)$ is analytic in a region $\boldsymbol{R}$ and on its boundary C and z_0 is an interior point of $\boldsymbol{R}$. Note that $g(z)$ is not analytic at $z = z_0$.



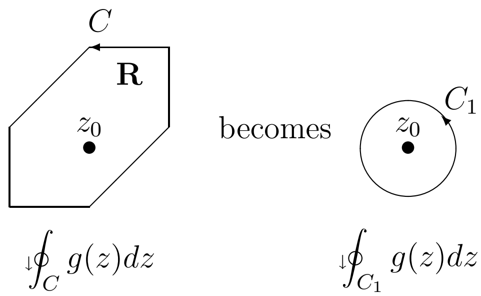
Consider



By Cauchy-Goursat

$$\downarrow\oint_C g(z)dz + \int_{L_1} g(z)dz + \uparrow\oint_{C_1} g(z)dz - \int_{L_1} g(z)dz = 0$$

$$\Rightarrow \downarrow\oint_C g(z)dz = \downarrow\oint_{C_1} g(z)dz.$$



On $C_1 : z = z_0 + \epsilon e^{i\theta} \Rightarrow dz = i\epsilon e^{i\theta} d\theta$.

$$\downarrow\oint_{C_1} g(z)dz = \downarrow\oint_{C_1} \frac{f(z)}{z - z_0} dz = \int_0^{2\pi} \frac{f\left(z_0 + \epsilon e^{i\theta}\right)}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta = \int_0^{2\pi} i \cdot f\left(z_0 + \epsilon e^{i\theta}\right) d\theta.$$

As $\epsilon \to 0$,

$$z_0 + \epsilon e^{i\theta} \to z_0 \Rightarrow f\left(z_0 + \epsilon e^{i\theta}\right) \to f\left(z_0\right) \text{ (a constant).}$$

$$\Rightarrow \downarrow\oint_{C_1} \frac{f(z)}{z - z_0} dz = f\left(z_0\right) \int_0^{2\pi} i d\theta = 2\pi i \cdot f\left(z_0\right)$$

or,

$$\downarrow\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i \cdot f\left(z_0\right)$$

This last result is the *Cauchy Integral Formula*.

Definition: $f\left(z_0\right)$ is called the *residue of* $g(z) = \frac{f(z)}{z - z_0}$ *at the point* $z = z_0$.

Note that

$$f\left(z_0\right) = \lim_{z \to z_0} g(z)\left(z - z_0\right).$$

Now

$$\downarrow\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i f\left(z_0\right).$$

Let $z = \xi,\ z_0 = z$. Then,

$$\downarrow\oint_C \frac{f(\xi)}{\xi - z} d\xi = 2\pi i f(z).$$

So,

$$f(z) = \frac{1}{2\pi i} \downarrow\!\oint_C \frac{f(\xi)}{\xi - z} d\xi.$$

Thus,

$$f'(z) = \frac{1}{2\pi i} \downarrow\!\oint_C \frac{f(\xi)}{(\xi - z)^2} d\xi$$

$$f''(z) = \frac{2}{2\pi i} \downarrow\!\oint_C \frac{f(\xi)}{(\xi - z)^3} d\xi$$

$$\vdots$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \downarrow\!\oint_C \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi$$

or back to original notation we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \downarrow\!\oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

Let

$$g(z) = \frac{f(z)}{(z - z_0)^m}.$$

Then,

$$\downarrow\!\oint_C g(z) dz = \downarrow\!\oint_C \frac{f(z)}{(z - z_0)^m} dz$$

and

$$\downarrow\!\oint_C \frac{f(z)}{(z - z_0)^m} dz = \frac{2\pi i}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} f(z) \big|_{z=z_0}$$

Definition: $\displaystyle \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \, g(z) \, (z - z_0)^m$ is called the *residue of*

$$g(z) = \frac{f(z)}{(z - z_0)^m}$$

at the point $z = z_0$, denoted Res $g(z) \big|_{z=z_0}$.

The above easily generalizes to the residue theorem.

Residue Theorem: Let $g(z)$ be a function which is analytic in a region R enclosed by the curve C except at some finite number of interior points: $z_0,\ z_1,\ \ldots,\ z_n$. Then,

$$\oint_C g(z)dz = 2\pi i \sum_{i=0}^{n} \text{Res } g(z)\ |_{z=z_i}$$

where,

$$\text{Res } g(z)\ |_{z=z_i} = \frac{1}{(m_i-1)!} \lim_{z\to z_i} \frac{d^{m_i-1}}{dz^{m_i-1}}\ g(z)\,(z-z_i)^{m_i}$$

and m_i is the power of $(z-z_i)$ in the expansion of $g(z)$ about z_i as a Laurent series, i.e., m_i is the order of the pole z_i.

Example: Evaluate

$$\oint_C \frac{z^2-1}{z^2+1}dz$$

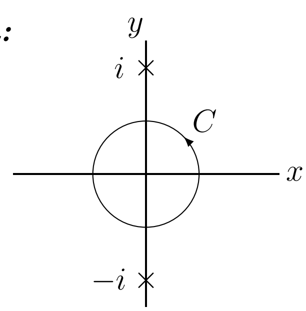
where C is

a. $|z| = 1/2$.

b. $|z-i| = 1$.

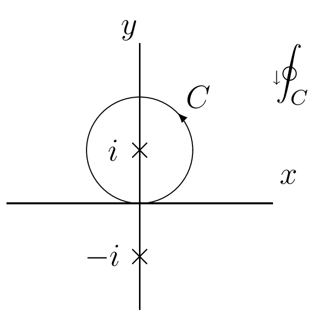
Solution:

a.



$$\oint_C \frac{z^2-1}{z^2+1}dz = 0$$

b.



$$\oint_C \frac{z^2-1}{z^2+1}dz = 2\pi i \lim_{z\to i} \frac{z^2-1}{z^2+1}(z-i) = -2\pi$$

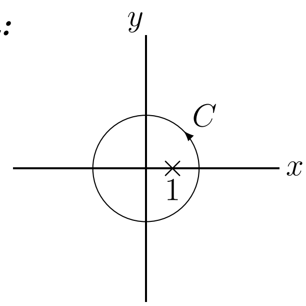
Example: Evaluate

$$\oint_C \frac{e^{2z}}{(z-1)^2} dz$$

where

$$C = \{z : |z| = 2\}.$$

Solution:



$$\oint_C \frac{e^{2z}}{(z-1)^2} dz = \frac{2\pi i}{(2-1)!} \lim_{z \to 1} \frac{d}{dz} \frac{e^{2z}}{(z-1)^2}(z-1)^2 = 2\pi i \left[2e^{2z} \mid_{z=1} \right] = 4\pi i e^2.$$

5.5 Inverse Z-transform

5.5.1 Direct Method

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Let C lie in the ROC of $X(z)$ and enclose the origin. So,

$$\frac{1}{2\pi i} \oint_C X(z) z^{k-1} dz = \frac{1}{2\pi i} \oint_C \sum_n x(n) z^{-n} z^{k-1} dz$$

$$= \sum_n x(n) \frac{1}{2\pi i} \oint_C z^{-n+k-1} dz.$$

Consider

$$I = \frac{1}{2\pi i} \oint_C z^{-n+k-1} dz.$$

Case 1: $n = k$.

$$I = \frac{1}{2\pi i} \oint_C z^{-1} dz = \frac{1}{2\pi i} 2\pi i = 1.$$

Case 2: $n < k \Rightarrow l = -n + k - 1 \geq 0$.

$$I = \frac{1}{2\pi i} \oint_C z^l dz = 0 \text{ by Cauchy-Goursat.}$$

Case 3: $n > k \Rightarrow -p = -n + k - 1 \leq -2$.

$$I = \frac{1}{2\pi i} \oint_C z^{-p} dz = \frac{1}{2\pi i} \frac{2\pi i}{(p-1)!} \lim_{z \to 0} \frac{d^{p-1}}{dz^{p-1}} \frac{1}{z^p} z^p$$

$$= \frac{1}{(p-1)!} \lim_{z \to 0} \frac{d^{p-1}}{dz^{p-1}} (1) = 0.$$

So,

$$\sum_n x(n) \frac{1}{2\pi i} \oint_C z^{-n+k-1} dz = x(k)$$

since

$$\frac{1}{2\pi i} \oint_C z^{-n+k-1} dz = \delta(n-k).$$

$$\Rightarrow x(k) = \frac{1}{2\pi i} {\downarrow}\oint_C X(z) z^{k-1} dz.$$

Thus, we have the inversion formula

$$x(n) = \frac{1}{2\pi i} {\downarrow}\oint_C X(z) z^{n-1} dz$$

or

$$x(n) = \sum_{\substack{\text{all poles} \\ \text{inside C}}} \text{Res } X(z) z^{n-1}.$$

Aside

Compute

$$I_0 = \int_0^\infty \frac{\sin x}{x} dx.$$

1. We will first use contour integration. Let

$$I = {\downarrow}\oint_C \frac{e^{iz}}{z} dz.$$

Then,

$$I_0 = \text{Im}\{I\}.$$

$$I = {\downarrow}\oint_C = \int_{C_1} + \int_{-R}^{-r} + \int_{C_2} + \int_r^R = 0 \text{ by Cauchy-Goursat.}$$

On C_1: $z = Re^{i\theta}$, $dz = iRe^{i\theta} d\theta$.

$$\left| \int_{C_1} \frac{e^{iz}}{z} \right| = \left| \int_0^\pi \frac{e^{iR(\cos\theta + i\sin\theta)}}{Re^{i\theta}} iRe^{i\theta} d\theta \right| \leq \int_0^\pi \left| ie^{iR\cos\theta} e^{-R\sin\theta} \right| d\theta$$

$$= \int_0^\pi e^{-R\sin\theta} d\theta \to 0$$

as $R \to \infty$ and $\theta \in (0, \pi)$ since $\sin\theta > 0$ in $(0, \pi)$. We can check the end points. For $\theta = 0$ or $\theta = \pi$, $e^{-R\sin\theta} = 1 < \infty$ so

$$\int_0^0 e^{-R\sin\theta} d\theta + \int_\pi^\pi e^{-R\sin\theta} d\theta = 0$$

thus, we conclude

$$\int_{C_1} \to 0$$

as $R \to \infty$ and $\theta \in [0, \pi]$.

On C_2: $z = re^{i\theta},\ dz = ire^{i\theta}d\theta$.

$$\int_{C_2} \frac{e^{iz}}{z}dz = \int_\pi^0 \frac{e^{ir(\cos\theta + i\sin\theta)}}{re^{i\theta}} ire^{i\theta}d\theta = \int_\pi^0 ie^{ir\cos\theta}e^{-r\sin\theta}d\theta \xrightarrow{r\to 0} \int_\pi^0 id\theta = -\pi i.$$

So,

$$\lim_{\substack{R\to\infty\\ r\to 0}} \int_{-R}^{-r} \frac{e^{iz}}{z}dz + \int_r^R \frac{e^{iz}}{z}dz = \pi i \Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{x}dx = \pi i.$$

Thus,

$$\int_{-\infty}^{\infty} \frac{\cos x}{x}dx + i\int_{-\infty}^{\infty} \frac{\sin x}{x}dx = \pi i$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x}{x}dx = 0 \text{ and } \int_{-\infty}^{\infty} \frac{\sin x}{x}dx = \pi$$

$$\Rightarrow \int_0^{\infty} \frac{\sin x}{x}dx = \pi/2.$$

2. We can also solve this integral using the Fourier method. Let

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt$$

so

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{i2\pi ft}df.$$

Define,

$$R_1(t) = \begin{cases} 1, & |t| \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Then,

$$\mathcal{F}\left[R_1(t)\right] = \frac{\sin 2\pi f}{\pi f}.$$

Thus,

$$\int_0^{\infty} \frac{\sin x}{x}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}dx = \frac{\pi}{2}\int_{-\infty}^{\infty} \frac{\sin 2\pi f}{\pi f}df$$

$$= \frac{\pi}{2}\int_{-\infty}^{\infty} \frac{\sin 2\pi f}{\pi f}e^{i2\pi ft}df\Big|_{t=0} = \frac{\pi}{2}\mathcal{F}^{-1}\left[\frac{\sin 2\pi f}{\pi f}\right]\Big|_{t=0}$$

$$= \frac{\pi}{2}R_1(t)\Big|_{t=0} = \pi/2.$$

However, contour integration can be used to evaluate integrals which otherwise might be difficult even with Fourier methods.

End Aside

5.5.2 Partial-Fraction Expansion

Here we will be concerned with rational z-transforms of the form

$$X(z) = \frac{N(z)}{D(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.$$

The roots of $D(z)$ are called *poles*. The roots of $N(z)$ are called *zeros*.

If $M < N$ and $a_N \neq 0$ then $X(z)$ is said to be *proper*.

Setting $N(z) = 0$ we get M roots: $c_i,\ i = 1, 2, \ldots, M$ (zeros).
Setting $D(z) = 0$ we get N roots: $d_j,\ j = 1, 2, \ldots, N$ (poles).

We can now write

$$X(z) = A \frac{\prod_{i=1}^{M} \left(1 - c_i z^{-1}\right)}{\prod_{j=1}^{N} \left(1 - d_j z^{-1}\right)}, \quad A = \frac{b_0}{a_0}.$$

$X(z) = 0$ at $z = c_i,\ i = 1, 2, \ldots, M$ (zeros).
$X(z) \to \infty$ at $z = d_j,\ j = 1, 2 \ldots, N$ (poles).
Each $(1 - c_i z^{-1})$ gives a pole at $z = 0$.
Each $(1 - d_j z^{-1})$ gives a zero at $z = 0$.
So, we get an additional M poles and N zeros at $z = 0$, so we have pole/zero cancellations.

If $M \geq N$, we get $(M - N)$ additional poles at $z = 0$.

If $M < N$ we get $(N - M)$ additional zeros at $z = 0$.
So, the total number of poles equals the total number of zeros.

Assume without loss of generality that $M \geq N$. Then,

$$X(z) = \frac{\sum_{k=0}^{M} b'_k z^{-k}}{\sum_{k=0}^{N} a'_k z^{-k}} = \underbrace{\sum_{k=0}^{M-N} c_k z^{-k}}_{X_1(z)} + \underbrace{\frac{\sum_{k=0}^{N-1} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \cdot \frac{z^N}{z^N}}_{X_2(z)=\frac{N(z)}{D(z)}}.$$

It follows that

$$x_1(n) = \begin{cases} c_n, & n = 0, 1, \ldots, M - N, \\ 0, & \text{elsewhere.} \end{cases}$$

or

$$x_1(n) = \sum_{k=0}^{M-N} c_k \delta(n - k).$$

Now

$$D(z) = a_0 \prod_{k=1}^{N} (z - d_k).$$

<u>Case 1</u>: Distinct poles, i.e., $d_i \neq d_j$ for $i \neq j$. Expand $\frac{X(z)}{z}$ as partial fractions.

$$\frac{X(z)}{z} = \frac{\sum_{k=0}^{N-1} b_k z^{N-k-1}}{\sum_{k=0}^{N} a_k z^{N-k}} = \sum_{k=1}^{N} \frac{A_k}{(z - d_k)}.$$

<u>Case 2</u>: Multiple-order poles. A pole p of order m contributes terms of the form

$$\frac{A_i}{(z - p)^i}, \quad i = 1, 2, \ldots, m.$$

<u>Note</u>: The inverse z-transform of a rational function $X(z)$ can also be obtained by long division to express it as a power series in z^{-1} (for a causal sequence or the causal part of a sequence) or as a power series in z (for a noncausal sequence or the noncausal part of a sequence).

5.5.3 Examples and Additional Results

i) $x(n) = \alpha^n u(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z) = \dfrac{z}{z-\alpha}, \quad \text{ROC} = \{z : |z| > |\alpha|\}.$
$x(n)$ is a right-sided sequence.

ii) $x(n) = -\alpha^n u(-n-1) \overset{\mathcal{Z}}{\longleftrightarrow} X(z) = \dfrac{z}{z-\alpha}, \quad \text{ROC} = \{z : |z| < |\alpha|\}.$
$x(n)$ is a left-sided sequence.

iii) $x(n) = \alpha^n u(n) + \beta^n u(-n-1) \overset{\mathcal{Z}}{\longleftrightarrow} X(z) = \dfrac{z}{z-\alpha} - \dfrac{z}{z-\beta},$

$$\text{ROC} = \{z : |\alpha| < |z| < |\beta|\}.$$

$x(n)$ is a two-sided sequence.

Example: Let

$$X(z) = \frac{z}{z-2} + \frac{z}{z+3}.$$

Find $x(n)$ for the following ROCs.

a. $|z| > 3$.

b. $|z| < 2$.

c. $2 < |z| < 3$.

d. $|z| > 2$.

Solution:

a. This is a right-sided sequence so

$$x(n) = 2^n u(n) + (-3)^n u(n).$$

b. This is a left-sided sequence so

$$x(n) = -2^n u(-n-1) - (-3)^n u(-n-1).$$

c. This is a two-sided sequence so

$$x(n) = 2^n u(n) - (-3)^n u(-n-1).$$

d. This is not a valid ROC since $\{z : |z| > 2\}$ includes a pole at $z = -3$, so corresponding $x(n)$ does not exist.

Let us now use residue theory on part (a) above.

$$X(z) = \frac{z}{z-2} + \frac{z}{z+3} = \frac{2z^2 + z}{(z-2)(z+3)}.$$

$$x(n) = \frac{1}{2\pi i}\oint_C \frac{2z^2+z}{(z-2)(z+3)} z^{n-1} dz = \frac{1}{2\pi i}\oint_C \frac{(2z+1)}{(z-2)(z+3)} z^n dz.$$

Here we have ROC$= \{z : |z| > 3\}$.

i. Consider $n \geq 0$. Then,

$$x(n) = \frac{2z+1}{(z-2)(z+3)} z^n (z-2)\Big|_{z=2} + \frac{2z+1}{(z-2)(z+3)} z^n (z+3)\Big|_{z=-3}$$

$$= 2^n + (-3)^n, \quad n \geq 0.$$

Note: For pole at $z = 2$,

$$x(n) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-2)} dz$$

where,

$$f(z) = \frac{(2z+1)z^n}{(z+3)}.$$

For pole at $z = -3$,

$$x(n) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z+3)} dz$$

where,

$$f(z) = \frac{(2z+1)z^n}{(z-2)}.$$

In either case, for $n \geq 0$, $f(z)$ is analytic at the pole in question.

ii. Consider $n < 0$. Let us look for example at the pole $z = 2$.

$$f(z) = \frac{(2z+1)z^n}{(z+3)} \rightarrow \text{ we get a pole at } z = 0 \text{ since } n < 0.$$

Let us now look at some specific values of n less than 0.

For $n = -1$ we get

$$x(-1) = \sum \text{Res} \frac{2z+1}{(z-2)(z+3)z} = \frac{5}{(5)(2)} + \frac{-5}{(-5)(-3)} + \frac{1}{(-2)(3)}$$
$$= 0.$$

For $n = -2$ we get

$$x(-2) = \sum \text{Res} \frac{2z+1}{(z-2)(z+3)z^2} = \frac{5}{(5)(4)} + \frac{-5}{(-5)(9)}$$

$$+ \frac{d}{dz} \frac{2z+1}{(z-2)(z+3)}\Big|_{z=0} = \frac{1}{4} + \frac{1}{9} + \frac{-12-1}{36}$$

$$= 0.$$

Similarly, $x(n) = 0 \; \forall \; n < 0$.

Thus, the end result is

$$x(n) = [2^n + (-3)^n] \, u(n).$$

The above residue method as given in the book requires specific calculations for $n < 0$ to see the trend. The following result avoids all of that.

General Residue Result: Put

$$X(z) = \frac{N(z)}{D(z)}$$

in proper fraction form. Then for the fractional part, $X_1(z)$, we have

$$x_1(n) = \sum_{\substack{\text{all poles} \\ \text{inside C}}} \text{Res } X_1(z) z^{n-1}, \quad m \geq 0$$

$$- \sum_{\substack{\text{all poles} \\ \text{outside C}}} \text{Res } X_1(z) z^{n-1}, \quad m < 0$$

where, m is the least degree of the numerator polynomial of $X_1(z)z^{n-1}$.

Note that in the above, finding $x(n)$ for the non-fractional part of $X(z)$ is easy.

Let us now apply this result to part (c) of our example.

$$X(z) = \frac{z}{z-2} + \frac{z}{z+3} = \frac{2z^2 + z}{(z-2)(z+3)}, \quad \text{ROC} = \{z : 2 < |z| < 3\}.$$

Dividing we get

$$X(z) = 2 + \frac{12 - z}{(z-2)(z+3)}.$$

So,

$$x(n) = 2\delta(n) + \mathcal{Z}^{-1}\left[\frac{12 - z}{(z-2)(z+3)}\right]$$

$$= 2\delta(n) + \sum_{\substack{\text{all poles} \\ \text{inside C}}} \text{Res } \frac{12 - z}{(z-2)(z+3)} z^{n-1}, \quad m \geq 0$$

$$- \sum_{\substack{\text{all poles} \\ \text{outside C}}} \text{Res } \frac{12 - z}{(z-2)(z+3)} z^{n-1}, \quad m < 0.$$

The numerator is

$$12z^{n-1} - z^n \Rightarrow m = n-1$$
$$m \geq 0 \Rightarrow n \geq 1$$
$$m < 0 \Rightarrow n \leq 0.$$

So,

$$x(n) = 2\delta(n) + \frac{12-z}{(z-2)(z+3)} z^{n-1}(z-2)\Big|_{z=2} u(n-1)$$
$$-\frac{12-z}{(z-2)(z+3)} z^{n-1}(z+3)\Big|_{z=-3} u(-n)$$

or

$$x(n) = 2\delta(n) + 2^n u(n-1) - (-3)^n u(-n).$$

Compare this result with that obtained with PFE:

$$x_1(n) = 2^n u(n) - (-3)^n u(-n-1).$$

Even those these two expressions look different you can check that $x_1(n) = x(n)\ \forall\ n$.

The next example shows we can obtain different representations for $x(n)$ even without using contour integration.

Example: Let

$$X(z) = \frac{1+z^{-1}}{1-1.5z^{-1}+0.5z^{-2}}, \quad \text{ROC} = \{z : |z| > 1\}.$$

i. Method 1.

$$X(z) = \frac{z^2+z}{z^2-1.5z+0.5} = 1 + \frac{2.5z-0.5}{z^2-1.5z+0.5}.$$

Now,

$$\frac{2.5z-0.5}{z^2-1.5z+0.5} = \frac{2.5z-0.5}{(z-1)(z-0.5)} = \frac{A}{z-1} + \frac{B}{z-0.5}.$$

$$A = \frac{2.5z - 0.5}{(z-1)(z-0.5)}(z-1)\Big|_{z=1} = 4$$

$$B = \frac{2.5z - 0.5}{(z-1)(z-0.5)}(z-0.5)\Big|_{z=0.5} = -1.5$$

so,

$$X(z) = 1 + \frac{4}{z-1} - \frac{1.5}{z-0.5} = 1 + 4z^{-1}\frac{z}{z-1} - 1.5z^{-1}\frac{z}{z-0.5}.$$

Thus,

$$x_1(n) = \delta(n) + \left[4 - 1.5(1/2)^{n-1}\right] u(n-1).$$

ii. Method 2.

$$X(z) = \frac{z^2 + z}{z^2 - 1.5z + 0.5} \Rightarrow \frac{X(z)}{z} = \frac{z+1}{z^2 - 1.5z + 0.5}.$$

Now,

$$\frac{z+1}{z^2 - 1.5z + 0.5} = \frac{4}{z-1} - \frac{3}{z-0.5}.$$

So,

$$X(z) = 4\frac{z}{z-1} - 3\frac{z}{z-0.5}.$$

Thus,

$$x_2(n) = [4 - 3(1/2)^n]\, u(n).$$

You can verify that $x_1(n) = x_2(n)\ \forall\ n$.

5.6 Transfer Functions of LTI Systems

Definition: $H(z) = \mathcal{Z}[h(n)]$ is called the *transfer function* of the LTI system, where $h(n)$ is the impulse response of the system.

Note: $y(n) = x(n) * h(n) \Rightarrow Y(z) = X(z)H(z)$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)}.$$

Example: Consider

$$y(n) - \frac{1}{2}y(n-1) = x(n), \quad y(-1) = 0.$$

Then,

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{z}{z - \frac{1}{2}}.$$

We have a pole at $z = 1/2$ and a zero at $z = 0$.

Now let

$$x(n) = u(n) \Rightarrow X(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}.$$

Then,

$$Y(z) = X(z)H(z) = \frac{z}{z-1} \cdot \frac{z}{z - \frac{1}{2}} = \frac{z^2}{(z-1)\left(z - \frac{1}{2}\right)}.$$

$$\frac{Y(z)}{z} = \frac{z}{(z-1)\left(z - \frac{1}{2}\right)} = \frac{2}{z-1} - \frac{1}{z - \frac{1}{2}}.$$

$$\Rightarrow Y(z) = 2\frac{z}{z-1} - \frac{z}{z - \frac{1}{2}}.$$

Thus, since our system is causal

$$y(n) = 2u(n) - \left(\frac{1}{2}\right)^n u(n).$$

Causality: Recall $h(n) = 0$ for $n < 0$ for a causal system. This type of system is a right-sided sequence. So, an LTI system is causal if and only if the ROC of $H(z)$ is ROC $= \{z : |z| > r\}$, some $r,\ 0 < r < \infty$.

BIBO Stability: Recall an LTI system is BIBO stable iff

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty.$$

Now,

$$H(z) = \sum_{n} h(n) z^{-n}.$$

So,

$$|H(z)| \leq \sum_{n} |h(n)| \left| z^{-n} \right|.$$

At $|z| = 1$, i.e., on the unit circle

$$|H(z)| \leq \sum_{n} |h(n)|.$$

This implies that an LTI system is BIBO stable iff the ROC includes the unit circle.

Example: Suppose the step response of a certain stable, causal LTI system is

$$s(n) = \left(\frac{1}{2}\right)^{n} u(n).$$

Find $h(n)$.

Solution:

$$x(n) = u(n) \Rightarrow y(n) = \left(\frac{1}{2}\right)^{n} u(n).$$

We have

$$X(z) = \frac{z}{z-1}, \quad Y(z) = \frac{z}{z-\frac{1}{2}}.$$

Thus,

$$H(z) = \frac{z-1}{z-\frac{1}{2}} = \frac{z}{z-\frac{1}{2}} - z^{-1} \frac{z}{z-\frac{1}{2}}.$$

We need the ROC of $H(z)$. The system is both causal and stable. Stability implies the ROC includes the unit circle. Causality implies the ROC is the exterior of a disc. So we have

$$\text{ROC} = \left\{ z : |z| > \frac{1}{2} \right\}.$$

Therefore,

$$h(n) = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{2}\right)^{n-1} u(n-1).$$

Example: Find the difference equation that corresponds to the transfer function

$$H(z) = \frac{z+1}{z^2+2z-3}.$$

Solution: We can write

$$H(z) = \frac{z^{-1}+z^{-1}}{1+2z^{-1}-3z^{-2}} = \frac{Y(z)}{X(z)}.$$

Then,

$$Y(z)\left[1+2z^{-1}-3z^{-2}\right] = X(z)\left[z^{-1}+z^{-2}\right].$$

Hence,

$$y(n)+2y(n-1)-3y(n-2) = x(n-1)+x(n-2).$$

5.7 Unilateral Z-transform

This transform will be useful to us for solving difference equations with initial conditions.

Definition: The *unilateral z-transform* is given by

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}.$$

Note: For causal sequences the unilateral z-transform is the same as the (bilateral) z-transform.

Shift Property:

$$\mathcal{Z}^+[x(n-k)] = \sum_{n=0}^{\infty} x(n-k)z^{-n} = \sum_{m=-k}^{\infty} x(m)z^{-(m+k)}$$

$$= z^{-k}\left[\sum_{m=-k}^{-1} x(m)z^{-m} + \sum_{m=0}^{\infty} x(m)z^{-m}\right]$$

$$= z^{-k}\left[\sum_{n=1}^{k} x(-n)z^{n} + \sum_{n=0}^{\infty} x(n)z^{-n}\right].$$

So,

$$x(n-k) \overset{\mathcal{Z}^+}{\longleftrightarrow} z^{-k}\left[X^+(z) + \sum_{n=1}^{k} x(-n)z^{n}\right], \quad k > 0.$$

Similarly,

$$x(n+k) \overset{\mathcal{Z}^+}{\longleftrightarrow} z^{k}\left[X^+(z) - \sum_{n=0}^{k-1} x(n)z^{-n}\right], \quad k > 0.$$

Example: Consider the difference equation

$$y(n) - \frac{1}{2}y(n-1) = x(n), \quad y(-1) = -2.$$

Find the complete solution when $x(n) = u(n)$.

Solution:

$$Y^+(z) - \frac{1}{2}z^{-1}\left[Y^+(z) + y(-1)z\right] = X^+(z).$$

Now,

$$X^+(z) = \frac{1}{1-z^{-1}}.$$

So,

$$Y^+(z) - \frac{1}{2}z^{-1}Y^+(z) + 1 = \frac{1}{1-z^{-1}}$$

$$Y^+(z)\left[1 - \frac{1}{2}z^{-1}\right] = \frac{1}{1-z^{-1}} - 1 = \frac{z^{-1}}{1-z^{-1}} = \frac{1}{z-1}.$$

Thus,

$$Y^+(z) = \frac{1}{(z-1)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{z}{(z-1)\left(z-\frac{1}{2}\right)} = \frac{2}{z-1} - \frac{1}{z-\frac{1}{2}}$$

$$= z^{-1}\frac{2z}{z-1} - z^{-1}\frac{z}{z-\frac{1}{2}}.$$

Therefore,

$$y(n) = \left[2 - \left(\frac{1}{2}\right)^{n-1}\right]u(n-1).$$

We could have proceeded as follows:

$$\frac{Y^+(z)}{z} = \frac{1}{(z-1)\left(z-\frac{1}{2}\right)} = \frac{2}{z-1} - \frac{2}{z-\frac{1}{2}}.$$

So,

$$Y^+(z) = \frac{2z}{z-1} - \frac{2z}{z-\frac{1}{2}}.$$

Hence,

$$y(n) = \left[2 - 2\left(\frac{1}{2}\right)^n\right] u(n)$$

which can be written (if we wish) as

$$y(n) = \left[2 - \left(\frac{1}{2}\right)^{n-1}\right] u(n-1)$$

to match our solution above.