3.0 System Properties

3.1 Definitions and Notations

Definition: A *discrete-time system* or just a *system* is any device or algorithm that performs operations on an input sequence to produce an output sequence.

Notation:

$$y(n) = T\left[x(n)\right].$$

Example:

a. $y(n) = x(n)$, the identity system.

b. $y(n) = x(n-1)$, the unit delay system.

c. $y(n) = x(n+1)$, the unit advance system.

d. $y(n) = \left[x(n) + x(n-1)\right]/2$, this system averages two successive samples of the input sequence.

Example:

$$y(n) = y(n-1) + x(n).$$

For this system suppose we apply an input sequence $x(n)$ for $n \geq 0$. Then,

$$y(0) = y(-1) + x(0).$$

We see that we need to know $y(-1)$ (the "initial condition") to determine the response of the system.

Definition: A *relaxed system* is that for which knowledge of the input sequence alone is sufficient to determine its response.

So, a relaxed system has an "initial state" of zero.

Block Diagrams In class we will discuss diagrams for adders, multipliers, the unit delay and unit advance operations.

Definition: A system is called *static* or *memoryless* if its output at time n depends at most on its input at time n.

Definition: A system is called *time-invariant* if a k-unit time delay in the input produces a k-unit time delay in the output, i.e.,

$$y(n) = T\left[x(n)\right] \Rightarrow y(n-k) = T\left[x(n-k)\right].$$

Otherwise, the system is called *time-variant.*

Example:

a. $y(n) = ax(n)$ is static and time-invariant.

b. $y(n) = x(n-1) + x(n)$ is not static but is time-invariant.

c. $y(n) = nx(n)$ is static and time-variant:

$$T\left[x(n)\right] = nx(n)$$

so,

$$T\left[x(n-k)\right] = nx(n-k)$$

but,

$$y(n-k) = (n-k)x(n-k) \neq T\left[x(n-k)\right].$$

Definition: A system is called *linear* if

$$ay_1(n) + by_2(n) = T\left[ax_1(n) + bx_2(n)\right] \;\; \forall \text{ constants } a, b$$

where,

$$y_i(n) = T\left[x_i(n)\right], \;\; i = 1, 2.$$

Otherwise, it is called *nonlinear.*

Definition: A system is said to be *causal* if its output at time n depends only on present and past values of the input sequence. Otherwise, it is said to be *noncausal.*

Example:

a. $y(n) = 2x(n) + x(n-1)$ is causal and linear.

b. $y(n) = x(n) + 1$ is causal and nonlinear:

$$T\left[x(n)\right] = x(n) + 1$$

so,

$$T\left[ax_1(n) + bx_2(n)\right] = ax_1(n) + bx_2(n) + 1$$

but,

$$ay_1(n) + by_2(n) = ax_1(n) + a + bx_2(n) + b \neq ax_1(n) + bx_2(n) + 1.$$

Definition: A system is said to be *bounded input-bounded output* (BIBO) *stable* if every bounded input sequence produces a bounded output sequence.

Note: $x(n)$ is a bounded sequence *iff* $\exists M \in \mathbf{Z}^+$ such that $|x(n)| \leq M \ \forall n$.

3.2 Delta and Unit Step Functions

3.2.1 Continuous Case

Definition: The *delta function*, $\delta(t)$, is a member of a class of generalized functions or distributions whose definition is in terms of its operation on a function inside an integral as

$$\int_{-\infty}^{\infty} \delta(t)dt = 1, \quad \delta(t) = 0 \ \forall t \neq 0$$

and

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

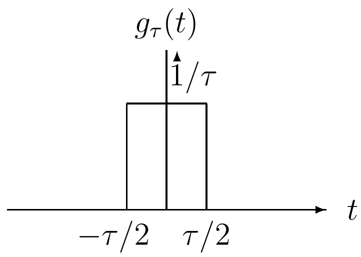
where $f(t)$ is an arbitrary function continuous a $t = 0$.

Note:

$$\int_{-\infty}^{\infty} f(t)\delta\left(t-t_0\right)dt = f\left(t_0\right)$$

if f is continuous at $t = t_0$.

One way the delta function can be visualized is



We take

$$\delta(t) = \lim_{\tau\to 0} g_\tau(t)$$

so

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = \lim_{\tau\to 0}\int_{-\tau/2}^{\tau/2} f(t)\frac{1}{\tau}dt \to f(0)\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} dt = f(0).$$

The above is valid since as $\tau \to 0$, $f(t) \to f(0)\ \forall t \in (-\tau/2, \tau/2)$.

Definition: The *unit step function*, $u(t)$, is given as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Let $g(t)$ be any function continuous at the origin with the constraint that $g(t) \to 0$ as $t \to \infty$ (a so called Class C function). Then,

$$\int_{-\infty}^{\infty} u'(t)g(t)dt = u(t)g(t)\mid_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(t)g'(t)dt$$

$$= -\int_{-\infty}^{\infty} u(t)g'(t)dt = -\int_{0}^{\infty} g'(t)dt = g(0) - g(\infty) = g(0).$$

So, here $u'(t)$ behaves like the delta function. Thus,

$$\frac{du(t)}{dt} = \delta(t)$$

when operating on functions of Class C. But, the delta function itself can be applied to a larger class of functions:

$$\int_{-\infty}^{\infty} g(t)\delta(t)dt = g(0)$$

only requires g continuous at $t = 0$; we do not need $g(t) \to 0$ as $|t| \to \infty$.

3.2.2 Discrete-time Case

Definition: The *unit sample sequence*, $\delta(n)$, is defined by

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

Definition: The *unit step signal*, $u(n)$, is defined by

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0. \end{cases}$$

These two function will play roles in the discrete case similar to the roles played by the corresponding functions in the continuous case.

3.3 Analysis of Discrete-time LTI Systems

Definition: The response of a linear time-invariant (LTI) system to the input sequence, $\delta(n)$, is called the *impulse response sequence* and is denoted by $h(n)$, i.e.,

$$h(n) = T\left[\delta(n)\right].$$

Now, any sequence, $x(n)$, can be written

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k).$$

The coefficients, $x(k)$, are simply the values of the samples of the sequence $x(n)$.

For an LTI system we have

$$y(n) = T[x(n)] = T\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] = \sum_{k=-\infty}^{\infty} x(k)T[\delta(n-k)]$$

$$= \sum_{k=-\infty}^{\infty} x(k)h(n-k).$$

Thus,

$$y(n) = x(n) * h(n).$$

Claim: $x(n) * h(n) = h(n) * x(n)$.

***Proof*:**

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \text{ [let } m = n-k]$$

$$= \sum_{m=n+\infty}^{n-\infty} x(n-m)h(m) = \sum_{m=-\infty}^{\infty} x(n-m)h(m) \text{ [let } k = m]$$

$$= \sum_{k=-\infty}^{\infty} h(k)x(n-k) = h(n) * x(n).$$

Claim: For causal LTI systems, $h(n) = 0$ for $n < 0$.

***Proof*:**

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{n} x(k)h(n-k) + \sum_{k=n+1}^{\infty} x(k)h(n-k).$$

Since our system is causal the output at time n does not depend on the input for $k > n$ so the right hand expression above must be zero. However, since we have no control over the input we must have

$$h(n-k) = 0 \ \forall k \geq n+1 \Rightarrow h(n) = 0 \ \forall n < 0.$$

Claim: An LTI system is BIBO stable *iff* $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$.

***Proof*:** "$\Leftarrow$" (if part or sufficiency).

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k).$$

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k)x(n-k) \right| \leq \sum_{k=-\infty}^{\infty} |h(k)| \, |x(n-k)| \, .$$

Since the input is bounded $\exists M$ such that $|x(n)| \leq M \; \forall n$

$$\Rightarrow |y(n)| \leq M \sum_{k=-\infty}^{\infty} |h(k)|.$$

So, if

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

then $|y(n)|$ is bounded.

***Proof*:** "$\Rightarrow$" (only if part or sufficiency). Want to show

$$|x(n)| < M_1 \Rightarrow |y(n)| < M_2$$

together imply

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

where, M_1 and M_2 are some integer bounds. It is enough to show that if $\sum_k |h(k)| = \infty$ then there exists a bounded input for which the output is not bounded (this is proof by contrapositive).

Let

$$x(-n) = \begin{cases} \dfrac{h^*(n)}{|h(n)|}, & h(n) \neq 0 \\ 0, & h(n) = 0. \end{cases}$$

Thus, $|x(-n)| = 1 \; \forall n$ so the input is bounded. We have

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) \frac{h^*(k-n)}{|h(k-n)|}.$$

Letting $n = 0$ we get

$$y(0) = \sum_{k=-\infty}^{\infty} h(k) \frac{h^*(k)}{|h(k)|} = \sum_{k=-\infty}^{\infty} \frac{|h(k)|^2}{|h(k)|}$$

$$= \sum_{k=-\infty}^{\infty} |h(k)| = \infty \text{ by assumption.}$$

So, $|x(n)| < M_1 \Rightarrow |y(n)| < M_2$ together imply $\sum_k |h(k)| < \infty$.

Example: Let

$$y(n) = \frac{1}{2}y(n-1) + x(n), \quad y(n) = 0 \ \forall n < 0.$$

a. Find $h(n)$.

b. Find $y(n)$ when $x(n) = u(n-1)$.

Solution:

a. Let $x(n) = \delta(n)$. Then,
$y(0) = 1$
$y(1) = 1/2 + 0 = 1/2$
$y(2) = 1/4$
$y(3) = 1/8$
etc.

Thus, $y(n) = (1/2)^n$ when $n \geq 0$ and $x(n) = \delta(n)$, so

$$h(n) = (1/2)^n u(n).$$

b. $y(n) = \sum_k x(k)h(n-k)$ so

$$y(n) = \sum_{k=-\infty}^{\infty} u(k-1)\left[(1/2)^{n-k}u(n-k)\right] = \sum_{k=1}^{\infty}(1/2)^{n-k}u(n-k)$$

$$= \sum_{k=1}^{n}(1/2)^{n-k} = (1/2)^n \sum_{k=1}^{n}(2)^k = (1/2)^n \frac{2-2^{n+1}}{1-2} = 2 - 2^{1-n}.$$

Therefore,

$$y(n) = \left[2 - (1/2)^{n-1}\right] u(n-1).$$

Useful Formula:

$$\sum_{n=M}^{N} a^n = \begin{cases} \dfrac{a^M - a^{N+1}}{1-a}, & a \neq 1, \\ N - M + 1, & a = 1. \end{cases}$$

Special Case: If $|a| < 1$ then

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}.$$

3.4 Difference Equations

3.4.1 Introduction

Consider

i. $y(n) = x(n) + x(n-1)$.

ii. $y(n) = y(n-1) + x(n)$.

Let us find the impulse response for these two systems.

i. $h(n) = \delta(n) + \delta(n-1)$.

$$\Rightarrow h(n) = \begin{cases} 1, & n = 0, 1, \\ 0, & \text{elsewhere}. \end{cases}$$

$h(n)$ is thus an example of a finite impulse response (FIR).

ii. $h(n) = h(n-1) + \delta(n)$.
So,
$h(0) = 1$.
$h(1) = h(0) + \delta(1) = 1$.
Similarly, $h(k) = 1, \ \forall k \geq 0$.
So, $h(n) = u(n)$.
$h(n)$ is thus an example of an infinite impulse response (IIR).

Block Diagrams An example of a feedforward and recursion system will be given in class. In general, LTI systems that are described by linear difference equations with constant coefficients are of the form

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k).$$

Note: There is often more than one way to realize a block diagram representation of these systems.

Definition: The realization of the system that requires the fewest number of delays is called *minimal.*

A theoretical approach that can be taken to design minimal systems may be found in texts on Linear Sequential Circuits. This analysis requires a comprehension of matrix algebra, group, ring and field theory.

3.4.2 Solving Difference Equations

Given a difference equation we would like to express $y(n)$ as a function of n. Let us write

$$y(n) = y_h(n) + y_p(n).$$

$y_h(n)$ is called the *homogeneous* solution and $y_p(n)$ is called the *particular* solution.

3.4.2.1 Homogeneous Solution

Here we assume $x(n) = 0$. We solve

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k)$$

or

$$\sum_{k=0}^{N} a_k y(n-k) = 0, \quad a_0 = 1.$$

Assume the solution is of the form $y_h(n) = \lambda^n$.

$$\Rightarrow \sum_{k=0}^{N} a_k \lambda^{n-k} = 0$$

$$\Rightarrow a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_N\lambda^{n-N} = 0$$

$$\Rightarrow \lambda^{n-N}\left(a_0\lambda^N + a_1\lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N\right) = 0.$$

The expression in parenthesis in the last equation is called the *characteristic polynomial*.

Note: There are N roots to the characteristic polynomial. The roots may be purely real but, in general, they are complex. Some roots may occur with multiplicity $m > 1$.

Distinct Roots Case

$$y_h(n) = c_1\lambda_1^n + c_2\lambda_2^n + \cdots + c_N\lambda_N^n.$$

The $c_i\ 's$ are constants to be determined using inital conditions.

Repeated Roots Case

Each repeated root, λ_i, with multiplicity m_i, contributes a term of the form

$$c_{1,i}\lambda_i^n + c_{2,i}n\lambda_i^n + c_{3,i}n^2\lambda_i^n + \cdots + c_{m_i,i}n^{m_i-1}\lambda_i^n$$

to the solution above for the distinct roots case. So, if k roots have multiplicity greater than 1, then

$$y_h(n) = \sum_{i=1}^{k}\sum_{j=1}^{m_i} c_{j,i}n^{j-1}\lambda_i^n + c_{k+1}\lambda_{k+1}^n + \cdots + c_N\lambda_N^n.$$

The justification for this general result follows along these lines. Define

$$p(\lambda) = \sum_{k=0}^{N} a_k\lambda^{n-k} = \sum_{k=0}^{N} a_k y(n-k) = 0, \quad \text{where } y(n) = \lambda^n.$$

If λ is a repeated root, say a double root, of $p(\lambda)$ then

$$p(\lambda) = p'(\lambda) = 0.$$

Now

$$p(\lambda) = \lambda^n \sum_{k=0}^{N} a_k\lambda^{-k} = \lambda^n q(\lambda)$$

and

$$p(\lambda) = 0 \;\Leftrightarrow\; q(\lambda) = 0.$$

$$p'(\lambda) = \sum_{k=0}^{N} a_k \lambda^{n-k-1}(n-k) = \lambda^n q'(\lambda) + n\lambda^{n-1} q(\lambda) = 0.$$

We observe that $\lambda^{n-k-1}(n-k) = y(n-k)$ as desired if $y(n) = n\lambda^{n-1}$. Thus,

$$y(n) = n\lambda^{n-1}$$

is also a solution.

Note: If $n = N$ we have a term in $p(\lambda)$ of the form $a_N \lambda^{N-N} = a_N$ and $\frac{d}{d\lambda}(a_N) = 0$, so when $n = N$

$$p'(\lambda) = \sum_{k=0}^{N-1} a_k \lambda^{n-k-1}(n-k).$$

But, $(n-k) = 0$ when $n = N,\ k = N$ thus

$$p'(\lambda) = \sum_{k=0}^{N} a_k \lambda^{n-k-1}(n-k)$$

is valid either way.

Remark

Say the characteristic equation is of degree 2 and has a repeated root. We just showed $n\lambda^{n-1}$ is also a solution. Thus,

$$y_h(n) = \hat{c}_1 \lambda^n + \hat{c}_2 n \lambda^{n-1}.$$

But above we claimed for a double root that

$$y_h(n) = c_1 \lambda^n + c_2 n \lambda^n.$$

Let us compare these two solutions at sample $n = n_0 > 1$ and see they are equivalent.

$$c_1 \lambda^{n_0} + c_2 n_0 \lambda^{n_0} = \hat{c}_1 \lambda^{n_0} + \hat{c}_2 n_0 \lambda^{n_0-1} \iff \lambda^{n_0}(c_1 - \hat{c}_1) + n_0 \lambda^{n_0-1}(c_2\lambda - \hat{c}_2) = 0.$$

Since n_0 is arbitrary and the constants are independent of n_0, we require

$$c_1 = \hat{c}_1 \text{ and } c_2\lambda = \hat{c}_2.$$

So,

$$y_h(n) = \hat{c}_1\lambda^n + \hat{c}_2 n\lambda^{n-1} = c_1\lambda^n + c_2\lambda n\lambda^{n-1} = c_1\lambda^n + c_2 n\lambda^n.$$

Example: Consider the difference equation

$$y(n) - y(n-1) - 2y(n-2) = 0.$$

Find the homogeneous solution.

Solution: The characteristic equation is

$$\lambda^2 - \lambda - 2 = 0$$

which has roots

$$\lambda_1 = 2, \quad \lambda_2 = -1.$$

So,

$$y_h(n) = c_1 2^n + c_2(-1)^n.$$

Example: Consider the difference equation

$$y(n) - 4y(n-1) + 4y(n-2) = 0.$$

Find the homogeneous solution.

Solution: The characteristic equation is

$$\lambda^2 - 4\lambda + 4 = 0$$

which has roots

$$\lambda_1 = 2, \quad \lambda_2 = 2.$$

So,

$$y_h(n) = c_1 2^n + c_2 n 2^n$$

3.4.2.2 Particular Solution

The particular solution satisfies the given difference equation. We assume for $y_p(n)$ a form that depends on $x(n)$.

$x(n)$	$y_p(n)$
$Au(n)$	$ku(n)$
$A\alpha^n u(n)$	$k\alpha^n u(n)$
$A\cos(\omega_0 n)$	$[k_1 \cos(\omega_0 n) + k_2 \sin(\omega_0 n)]\, u(n)$

See text for others.

Example: Consider the difference equation

$$y(n) + 2y(n-1) - 8y(n-2) = 2x(n-1), \quad x(n) = (1/2)^n u(n).$$

Find the particular solution.

Solution: Assume

$$y_p(n) = k(1/2)^n u(n).$$

We get

$$k(1/2)^n u(n) + 2k(1/2)^{n-1} u(n-1) - 8k(1/2)^{n-2} u(n-2)$$
$$= 2(1/2)^{n-1} u(n-1).$$

We evaluate this expression at any n for which none of the terms vanish. Using $n = 2$,

$$k/4 + k - 8k = 1 \Rightarrow k = -4/27.$$

So,

$$y_p(n) = -\frac{4}{27}(1/2)^n, \; n \geq 2.$$

3.4.2.3 Complete Solution

The complete solution can be determined as follows:

1. Determine the particular solution, $y_p(n)$, as above.
2. Determine the homogeneous solution, $y_h(n)$, with the constants left unspecified.

3. The complete response is the sum of the homogeneous solution and the particular solution.

4. If necessary, propagate the initial conditions up to the time instant in which the particular response becomes valid.

5. Determine the constants from these initial conditions of step 4.

6. The resulting solution is the complete solution.

Example: Consider the difference equation

$$y(n) + 2y(n-1) - 8y(n-2) = 2x(n-1), \quad y(-1) = 1,\ y(-2) = 0,$$

$$x(n) = (1/2)^n u(n).$$

Find the complete solution.

Solution: We get

1. $y_p(n) = -\frac{4}{27}(1/2)^n,\ n \geq 2.$

2. The characteristic equation is

$$\lambda^2 + 2\lambda - 8 = 0 \Rightarrow \lambda_1 = 2,\ \lambda_2 = -4.$$

$$\Rightarrow y_h(n) = c_1 2^n + c_2(-4)^n,\ n \geq 0.$$

3. We now have

$$y(n) = c_1 2^n + c_2(-4)^n - \frac{4}{27}(1/2)^n.$$

4. We need $y(0)$ and $y(1)$.

$$y(n) + 2y(n-1) - 8y(n-2) = 2x(n-1)$$

$$\Rightarrow y(n) = -2y(n-1) + 8y(n-2) + 2x(n-1)$$

$$\Rightarrow y(0) = -2y(-1) + 8y(-2) + 2x(-1) = -2$$

$$\Rightarrow y(1) = -2y(0) + 8y(-1) + 2x(0) = 4 + 8 + 2 = 14.$$

5. We solve for the constants:

$$y(0) = -2 = c_1 + c_2 - \frac{4}{27}$$

$$y(1) = 14 = 2c_1 - 4c_2 - \frac{2}{27}$$

$$\Rightarrow c_1 = \frac{10}{9},\ c_2 = -\frac{80}{27}.$$

6. The complete solution is

$$y(n) = \frac{10}{9}2^n - \frac{80}{27}(-4)^n - \frac{4}{27}(1/2)^n,\ n \geq 0.$$

3.4.2.4 Impulse Response

Since here $x(n) = \delta(n)$ is of finite duration, we do not have a particular solution as when $x(n)$ is of infinite duration. Instead, we solve for the homogeneous solution with the constants accounting for $\delta(n)$. This is best illustrated with an example.

Example: We are given the difference equation

$$y(n) + 2y(n-1) - 8y(n-2) = 2x(n), \quad y(-1) = y(-2) = 0.$$

We wish to find the impulse response $h(n)$.

Solution: Let $x(n) = \delta(n)$. For $n = 0$ we have

$$y(0) + 2y(-1) - 8y(-2) = 2 \Rightarrow y(0) = 2.$$

The system is now unforced for $n > 0$, i.e., $n \geq 1$. We use $y(0)$ and $y(-1)$ as initial conditions. The characteristic equation is

$$\lambda^2 + 2\lambda - 8 = 0 \Rightarrow \lambda_1 = 2,\ \lambda_2 = -4.$$

So,

$$y_h(n) = c_1 2^n + c_2(-4)^n.$$

$$y(-1) = c_1/2 - c_2/4 = 0$$

$$y(0) = c_1 + c_2 = 2$$
$$\Rightarrow c_1 = \frac{2}{3},\ c_2 = \frac{4}{3}.$$
$$\Rightarrow h(n) = \frac{2}{3}2^n + \frac{4}{3}(-4)^n,\ n \geq 0.$$

Example: We are given the difference equation

$$y(n) + 2y(n-1) - 8y(n-2) = x(n) - 2x(n-1), \quad y(-1) = y(-2) = 0.$$

Find $h(n)$.

Solution: Using $x(n) = \delta(n)$ we see the system becomes unforced at $n = 2$. So we must determine the initial conditions at time $n = 0$ and $n = 1$.

$$y(0) + 2y(-1) - 8y(-2) = \delta(0) - 2\delta(-1) \Rightarrow y(0) = 1.$$

$$y(1) + 2y(0) - 8y(-1) = \delta(1) - 2\delta(0) \Rightarrow y(1) + 2y(0) = -2 \Rightarrow y(1) = -4.$$

The homogeneous solution for this new system is

$$y_h(n) = c_1 2^n + c_2(-4)^n$$

$$y(0) = c_1 + c_2 = 1$$
$$y(1) = 2c_1 - 4c_2 = -4$$
$$\Rightarrow c_1 = 0,\ c_2 = 1.$$

So, the impulse response is

$$h(n) = (-4)^n,\ n \geq 0.$$

Example Equation: In some of the following we shall refer to the following

difference equation (denoted by *)

$$y(n) + 2y(n-1) - 8y(n-2) = 2x(n-1), \qquad (*)$$

$$y(-1) = 1,\ y(-2) = 0,\ x(n) = (1/2)^n u(n).$$

We get

$$y(n) = c_1 2^n + c_2(-4)^n - \frac{4}{27}(1/2)^n$$

as the complete response.

3.4.2.5 Forced or Zero-State Response

Definition: The *forced response* or *zero-state response* of the system is defined as the complete response that follows by assuming zero initial conditions.

In (*), let $y(-1) = y(-2) = 0$

$$\Rightarrow y(0) = 0 = c_1 + c_2 - \frac{4}{27}$$

$$\Rightarrow y(1) + 2y(0) = 2 \Rightarrow y(1) = 2 = 2c_1 - 4c_2 - \frac{2}{27}$$

$$\Rightarrow c_1 = \frac{12}{27},\ c_2 = -\frac{8}{27}.$$

Thus,

$$y_{zs}(n) = \frac{12}{27}2^n - \frac{8}{27}(-4)^n - \frac{4}{27}(1/2)^n,\ n \geq 0.$$

3.4.2.6 Zero-Input Response

Definition: The *zero-input response* or *natural response* or *unforced response* of the system is defined as the homegeneous solution with the constants determined from the initial conditions.

Using (*) we get

$$y_h(n) = c_1 2^n + c_2(-4)^n.$$

Using $y(-1) = 1,\ y(-2) = 0$ we get

$$y(-1) = 1 = c_1/2 - c_2/4$$

$$y(-2) = 0 = c_1/4 + c_2/16$$

$$\Rightarrow c_1 = \frac{2}{3},\ c_2 = -\frac{8}{3}.$$

Thus,

$$y_{zi}(n) = \frac{2}{3}2^n - \frac{8}{3}(-4)^n,\ n \geq 0.$$

Note that the complete response is

$$y(n) = y_{zi}(n) + y_{zs}(n).$$

For our system in (*) we get the complete response as

$$y(n) = \frac{10}{9}2^n - \frac{80}{27}(-4)^n - \frac{4}{27}(1/2)^n, \ n \geq 0.$$

3.4.2.7 Transient Response

Definition: The *transient response* is the zero-input response when the roots of the characteristic equation all have magnitude less than 1, i.e., $|\lambda_i| < 1 \, \forall \lambda_i$.

The transient response decays to zero as $n \to \infty$.

3.4.2.8 Steady-State Response

Definition: If the zero-input response is transient, then

$$\lim_{n \to \infty} y(n) = y_{zs}(n)$$

and $y_{zs}(n)$ is also known as the *steady-state response*.

3.4.2.9 Step Response

Let $x(n) = u(n)$. Then,

$$y(n) = \sum_{k=-\infty}^{\infty} u(k)h(n-k) = \sum_{k=0}^{\infty} h(n-k).$$

Definition: The *step response* is defined as

$$s(n) = \sum_{k=0}^{\infty} h(n-k).$$

So, if you have the impulse response you can use it to find the step response.

3.5 Correlation of Discrete-time Signals

The idea behind correlation is to measure the similarity between signals or of one signal with a shifted version of itself. Computationally, correlation is similar to convolution.

Definition: The *cross-correlation* of two sequences $x(n)$ and $y(n)$ is

$$r_{xy}(n) = \sum_{k=-\infty}^{\infty} x(k)y(k-n), \quad n = 0, \pm 1, \pm 2, \ldots .$$
$$= \sum_{k=-\infty}^{\infty} y(k)x(k+n).$$

Recall,

$$x(n) * y(n) = \sum_{k=-\infty}^{\infty} x(k)y(n-k).$$

Replacing $y(n-k)$ by $y\left(-(n-k)\right)$ gives us $r_{xy}(n)$. Thus,

$$r_{xy}(n) = x(n) * y(-n).$$

Definition: The *auto-correlation* of the sequence $x(n)$ is

$$r_{xx}(n) = \sum_{k=-\infty}^{\infty} x(k)x(k-n), \quad n = 0, \pm 1, \pm 2, \ldots .$$

Notes

i. $r_{xx}(n) = r_{xy}(n)$ when $y(n) = x(n)$.

ii. $r_{xx}(0) = \sum_k x^2(k) = E_x =$ energy.

iii. $r_{xx}(0) \geq r_{xx}(n)\ \forall n$. See text on pp. 122-123 for a proof.

Definition: The *normalized auto-correlation sequence* is

$$\rho_{xx}(n) = \frac{r_{xx}(n)}{r_{xx}(0)}, \quad -1 \leq \rho_{xx}(n) \leq 1,$$

and the *normalized cross-correlation sequence* is

$$\rho_{xy}(n) = \frac{r_{xy}(n)}{\sqrt{r_{xx}(0)r_{yy}(0)}}, \quad -1 \leq \rho_{xy}(n) \leq 1.$$