11.0 Filter Design

The idea is to design a digital filter to approximate some desired response.

11.1 FIR Filter Design

i. Least Squares Method

Given a desired response

$$H_d(\omega) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-i\omega n}$$

and

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) e^{i\omega n} d\omega$$

then we can minimize

$$\epsilon = \int_{-\pi}^{\pi} \left| H(\omega) - H_d(\omega) \right|^2 d\omega$$

(where $h(n) \longleftrightarrow H(\omega)$ and $h(n)$ is of length N)

by letting

$$h(n) = \begin{cases} h_d(n), & |n| \le M, \; M = \frac{N-1}{2} \\ 0, & \text{elsewhere.} \end{cases}$$

To see this note that minimizing

$$\epsilon = \int_{-\pi}^{\pi} \left| H(\omega) - H_d(\omega) \right|^2 d\omega$$

is equivalent to minimizing

$$\epsilon = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| H(\omega) - H_d(\omega) \right|^2 d\omega.$$

By Parseval's relation we have

$$\epsilon = \sum_{n=-\infty}^{\infty} \left| h(n) - h_d(n) \right|^2$$

which we may write as

$$\epsilon = \sum_{n=-M}^{M} |h(n) - h_d(n)|^2 + \sum_{n=M+1}^{\infty} |h_d(n)|^2 + \sum_{n=-\infty}^{-M-1} |h_d(n)|^2$$

or

$$\epsilon = \sum_{n=-M}^{M} |h(n) - h_d(n)|^2 + 2 \sum_{n=M+1}^{\infty} |h_d(n)|^2$$

where we have assumed we have even symmetry with N odd and $M = \frac{N-1}{2}$. A similar procedure can be used for other types.

Therefore, the least squares solution is

$$h(n) = \begin{cases} h_d(n), & |n| \leq M, \; M = \frac{N-1}{2} \\ 0, & \text{elsewhere.} \end{cases}$$

Using

$$w(n) = \begin{cases} 1, & |n| \leq M, \\ 0, & \text{elsewhere} \end{cases}$$

then,

$$h(n) = h_d(n)w(n).$$

$w(n)$ is a rectangular window. We can use other windows to lessen certain effects (such as Gibbs phenomenon) at the expense of worsening other effects (such as main lobe width). With the least squares method we do not have control over the response at critical frequencies.

So,

$$H(\omega) = \sum_{n=-M}^{M} h_d(n)e^{-i\omega n} = h_d(0) + 2\sum_{n=1}^{M} h_d(n)\cos(\omega n).$$

A causal solution with the same magnitude response and linear phase is

$$h(n) = h_d(n - M), \quad n = 0, 1, \ldots, N-1.$$

ii. Frequency-Sampling Method
Here we specify the desired frequency response, $H_d(\omega)$, at a set of equally spaced frequencies to get $H_d(k)$ and then take the inverse DFT to obtain $h_d(n)$.

iii. Equiripple Design Method
This method will be discussed in class.

Example: *Design of a low pass FIR filter using least squares.* Let the filter length be $N = 13$. We want to approximate an ideal LPF with cutoff frequency $f_c = 100$ Hz for a sampling frequency of $f_s = 1000$ Hz. Since f_s corresponds to 2π we have $\omega_c = 2\pi(f_c/f_s) = \pi/5$. The desired response is

$$H_d(\omega) = \begin{cases} 1, & |\omega| \leq \pi/5, \\ 0, & \text{elsewhere} \end{cases}$$

Therefore,

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi/5}^{\pi/5} e^{i\omega n} d\omega = \frac{1}{5} \cdot \frac{\sin(n\pi/5)}{n\pi/5} = \frac{1}{5} \cdot \text{sinc}(n\pi/5).$$

For a causal solution we let

$$h(n) = h_d(n-6), \quad n = 0, 1, 2, \ldots, 12.$$

This gives

$$\underline{h} = \left[\frac{1}{5}\text{sinc}\left(-\frac{6\pi}{5}\right), \ \frac{1}{5}\text{sinc}\left(-\pi\right), \ldots, \frac{1}{5}\text{sinc}\left(\frac{6\pi}{5}\right)\right].$$

The resulting frequency response is

$$H(\omega) = e^{-i6\omega} \sum_{n=-6}^{6} \frac{1}{5}\text{sinc}\left(\frac{n\pi}{5}\right) e^{-i\omega n}.$$

11.2 Analog Filter Design

One way to design a digital filter is to design an analog filter and apply a transformation.

i. Butterworth filters have a magnitude response which is maximally flat near $\Omega = 0$ and declines monotonically for $\Omega > 0$.

ii. Chebyshev filters have a magnitude response which exhibits equal ripple for frequencies in the passband and then decline monotonically.

iii. Elliptic filters exhibit equal ripple in both the passband and stopband.

These filters are characterized by

$$|H(\Omega)|^2 = H(s)H(-s)\Big|_{s=i\Omega}$$

where $H(s)$ has poles and zeros in the left half plane and $H(-s)$ has poles and zeros in the right half plane.

Butterworth Filters

A Butterworth filter of order n is characterized by

$$|H(\Omega)|^2 = \frac{1}{1+\Omega^{2n}}.$$

For $\Omega^2 < 1$, we get

$$|H(\Omega)|^2 = 1 - \Omega^{2n} + \Omega^{4n} - \cdots.$$

Thus, the derivatives of $|H(\Omega)|^2$ with respect to Ω vanish at $\Omega = 0$ for orders up to $2n-1$. It is maximally flat in the passband.

Now,

$$|H(\Omega)|^2 = 1 - \epsilon \Rightarrow \Omega_p = \left(\frac{\epsilon}{1-\epsilon}\right)^{1/2n}$$

and

$$|H(\Omega)|^2 = \delta \Rightarrow \Omega_s = \left(\frac{1-\delta}{\delta}\right)^{1/2n}.$$

If $0 < \epsilon, \delta < 1/2$ then $\Omega_p \to 1, \Omega_s \to 1$ as $n \to \infty$.

The zeros of Butterworth filters are all at infinity (all pole filter). If we let $\Omega = s/i$ we see the poles satisfy

$$1 + (-is)^{2n} = 0.$$

Keeping the roots of this polynomial that lie in the left half plane we find the poles to be

$$\lambda_k = e^{i\theta_k}, \quad \theta_k = \frac{\pi}{2}\left(1 + \frac{2k-1}{n}\right), \quad 1 \le k \le n.$$

11.3 IIR Filter Design

11.3.1 Impulse Invariance Approach

Find $h_a(t) \longleftrightarrow H_a(\Omega)$ of analog system which meets desired digital response specification for sample rate T. Let $h(n) = h_a(nT)$. Then,

$$H(\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a(i\omega/T + i2\pi k/T).$$

We have aliasing since the analog filter (eg., Butterworth) response only goes to zero as $\Omega \to \infty$ so this approach is not good for high pass filters and should only be used for filters that are narrowband.

Now let $s = i\Omega = i\omega/T$. Then,

$$H(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a(s + i2\pi k/T).$$

Consider the partial fraction expansion of $H_a(s)$ as

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{(s - s_k)}.$$

Then,

$$h_a(t) = \sum_{k=1}^{N} A_k e^{s_k t} u(t)$$

and

$$h(n) = \sum_{k=1}^{N} A_k e^{s_k nT} u(n).$$

Thus,

$$H(z) = \sum_{n=0}^{\infty} \sum_{k=1}^{N} A_k e^{s_k nT} z^{-n}$$

$$= \sum_{k=1}^{N} A_k \cdot \frac{1}{1 - e^{s_k T} z^{-1}}$$

$$\Rightarrow H(z) \text{ has poles } z_k = e^{s_k T}.$$

11.3.2 Bilinear Transform Approach

Apply transform $\Omega = c \cdot \tan(\omega/2)$. "$c$" is a constant and depends on the filter. Now

$$i\Omega = ic \left(\frac{e^{i\omega/2} - e^{-i\omega/2}}{e^{i\omega/2} + e^{-i\omega/2}} \right) \frac{1}{i}$$

$$= c \cdot \frac{e^{i\omega} - 1}{e^{i\omega} + 1}$$

or

$$s = c \cdot \frac{z-1}{z+1}.$$

Let us now consider the choice of "c" for a Butterworth filter. Here

$$|H(\Omega)|^2 = \frac{1}{1 + \Omega^{2n}}.$$

We get

$$|H(\Omega)|^2\Big|_{\Omega=1} = \frac{1}{2}.$$

Here we assume that things are normalized so that $|H(0)| = 1$. If this is not the case then divide $H(\Omega)$ by $|H(0)|$. Now

$$\Omega = c \cdot \tan(\omega/2).$$

$\Omega_c = 1$, $\omega_c = -3$ dB frequency. Thus, $1 = c \cdot \tan(\omega_c/2)$. Therefore,

$$c = \cot(\omega_c/2).$$