10.0 Quantization Effects

10.1 Coefficient Quantization

Assume b-fractional bits are available to represent a coefficient. So, $LSB = 2^{-b}$. Let α equal the coefficient. Then, with quantization we get

$$Q[\alpha] = L \cdot 2^{-b}, \ L \text{ an integer.}$$

Choose L to minimize $|\alpha - L \cdot 2^{-b}|$.

Consider the first order case

$$H(z) = \frac{1}{1 - \alpha z^{-1}}.$$

We have a pole at $z = \alpha$. Suppose α is real. Then,

$$\Delta Q[\alpha] = 2^{-b}.$$

10.2 Rounding Errors in Signals

$$Q[x(n)] = x(n) + e_q(n)$$

where, $x(n)$ represents the perfect signal and $e_q(n)$ here represents the error "signal" due to quantization. Suppose we round the signal to b-bits. Assume that $|x(n)| < 1$. Then, $LSB = 2^{-b}$. We may need an additional bit for the sign bit so the total number of bits is $b + 1$.

The rounding error is

$$|e_q(n)| \leq \frac{2^{-b}}{2}.$$

In terms of random variables e_q is uniform between $-\frac{2^{-b}}{2}$ and $\frac{2^{-b}}{2}$, i.e.,

$$e_q \sim U\left[-\frac{2^{-b}}{2}, \ \frac{2^{-b}}{2}\right)$$

so

$$E\left[e_q\right] = \int_{-\frac{2^{-b}}{2}}^{\frac{2^{-b}}{2}} e_q \cdot 2^b de = 0$$

and

$$E\left[e_q^2\right] = \int_{-\frac{2^{-b}}{2}}^{\frac{2^{-b}}{2}} e_q^2 \cdot 2^b de = \frac{2^{-2b}}{12}.$$

Thus,

$$Var\left[e_q\right] = E\left[e_q^2\right] - E^2\left[e_q\right] = \frac{2^{-2b}}{12}.$$

10.3 SNR in Sampled Signals

Say the A/D has b bits $\Rightarrow 2^b$ levels. Let us look at

$$Q[x(n)] = x(n) + e_q(n).$$

We have

$$SQNR_{dB} = 10\log_{10}\left[\frac{P_s}{P_{qn}}\right] \text{ (dB)}$$

where, P_s is the power in the signal and P_{qn} is the power in the quantization noise.

For quantization noise only we get

$$SQNR_{dB} = 10\log_{10}\left[\frac{P_s}{2^{-2b}/12}\right]$$

or

$$SQNR_{dB} = 6.02b + 10.79 + 10\log_{10} P_s.$$

From this last expression we see that for each extra bit in the A/D converter we get 6.02 dB improvement in $SQNR_{dB}$.

Note that the above assumed $|x(n)| < 1$. This is no loss in generality since we can always assume the incoming data is scaled so as to be a fraction.

If the input signal $x(t)$ has random noise then $x(n)$ also has random noise even if the quantization noise is zero. The above analysis is for the performance of the A/D relative to its input whether the input is clean or has noisy data.

The total signal-noise-ratio is

$$SNR_{dB} = 10\log_{10}\left[\frac{P_s}{P_n + P_{qn}}\right] \text{ (dB)}$$

where, P_n is the power in the thermal noise.

10.4 Noise Propagation

If

$$Q[x(n)] = x(n) + e_q(n).$$

then the output is $y(n) + f(n)$ where,

$$y(n) = x(n) * h(n), \quad f(n) = e_q(n) * h(n).$$

Output Noise Variance

$$E\left[e_q(n)\right] = 0, \quad E\left[(e_q(n))^2\right] = \frac{2^{-2b}}{12}, \quad E[e_q(n)e_q(m)] = 0, \ n \neq m.$$

Also,

$$Var[e_q(n)] = \sigma_{e_q}^2 = E\left[(e_q(n))^2\right] - E^2\left[e_q(n)\right] = \frac{2^{-2b}}{12}.$$

Now assume $x(n)$ and $e_q(n)$ are statistically independent. Then,

$$E[f(n)] = E\left[\sum_{k=0}^{\infty} h(k)e_q(n-k)\right] = \sum_{k=0}^{\infty} h(k)E\left[e_q(n-k)\right] = 0.$$

Note that expectation is a linear operator.

Also,

$$\sigma_f^2 = E[(f(n))^2] - E^2[f(n)] = E\left[\sum_{k=0}^{\infty} h(k)e_q(n-k) \cdot \sum_{l=0}^{\infty} h(l)e_q(n-l)\right]$$

$$= \sum_k \sum_l h(k)h(l)E\left[e_q(n-k)e_q(n-l)\right]$$

$$= \sum_k \sum_l h(k)h(l)\delta(k-l)\sigma_{e_q}^2$$

or

$$\sigma_f^2 = \sigma_{e_q}^2 \sum_{k=0}^{\infty} (h(k))^2$$

in the real case and

$$\sigma_f^2 = \sigma_{e_q}^2 \sum_{k=0}^{\infty} |h(k)|^2$$

in the complex case.

Example: Let

$$H(z) = \frac{1}{1-\alpha z^{-1}}, \quad 0 < |\alpha| < 1.$$

Then,

$$h(n) = \alpha^n u(n).$$

The output noise variance is

$$\sigma_f^2 = \sigma_{e_q}^2 \sum_{n=0}^{\infty} \alpha^{2n}$$

or

$$\sigma_f^2 = \frac{2^{-2b}}{12} \cdot \frac{1}{1-\alpha^2}.$$

We see that as b increases, σ_f^2 decreases and as $\alpha \to 1$, σ_f^2 increases.