# EE 113

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## Lecture Notes

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# 1.0 Signals and Systems

**Definition:** A *signal* is a function of one or more independent variables.

A function $x$ is defined by

$$x : T \rightarrow \Omega$$

denoted by $x(\tau) : \tau \in T$. $T$ denotes the domain of the function and $\Omega$ denotes the range of the function.

## 1.1 Continuous Time Signals

Let $\mathbf{R}$ denote the real line: $(-\infty, \infty)$.

Let $\mathbf{Z}$ denote the set of integers: $0, \pm 1, \pm 2, \ldots$.

An *analog signal* has $T = \mathbf{R}$, $\Omega = \mathbf{R}$.

A *quantized analog signal* has $T = \mathbf{R}$, $\Omega = \mathbf{Z}$.

## 1.2 Discrete-Time and Digital Signals

A *discrete-time signal* has $T = \mathbf{Z}$, $\Omega = \mathbf{R}$. We will also call this type of signal a *sequence*.

A *digital signal* has $T = \mathbf{Z}$, $\Omega = \mathbf{Z}$.

**Remarks**

1. An analog computer processes analog signals.
2. A digital computer processes digital signals.
3. Our main concern is with digital signal processing but we will usually analyze signals as if they were discrete-time signals because more mathematical tools are available for discrete-time analysis. We can quantify the resulting error (quantization effects).

4. For images we have

   i. $T = \mathbf{R} \times \mathbf{R} = \mathbf{R}^2$, $\Omega = \mathbf{R}$ or $\mathbf{Z}$ for continuous images.

   ii. $T = \mathbf{Z} \times \mathbf{Z} = \mathbf{Z}^2$, $\Omega = \mathbf{R}$ or $\mathbf{Z}$ for discrete images. In this case $\tau = (m, n),\ m \in \mathbf{Z}, n \in \mathbf{Z}$.

## 1.3 Applications of DSP

- Frequency and Spectral Analysis
- Filter Design and Implementation
- Adaptive Filtering (eg., noise cancelling and channel equalization)
- Signal Detection
- System Identification
- Radar and Sonar Operations
- Array Processing
- Speech Processing
- Image Processing
- Pattern Recognition
- Digital Communications
- Medical Applications
- Many Others

## 1.4 Concept of Frequency

### 1.4.1 Continuous Case

Consider

$$x_a(t) = A\cos(\Omega t + \theta),\ t \in \mathbf{R}$$

- $A$ is the amplitude
- $\Omega$ is the frequency (radians per second)
- $t$ is time (seconds)
- $\theta$ is the phase (radians).

We also write $\Omega = 2\pi F$, $F$ is frequency (cycles per second or Hertz). So,

$$x_a(t) = A\cos(2\pi F t + \theta),\ t \in \mathbf{R}.$$

The *period* of this sinusoid is $T_p = 1/F$ since

$$x_a(t + 1/F) = A\cos(2\pi F(t + 1/F) + \theta) = A\cos(2\pi F t + \theta).$$

We observe that as $F$ increases then $T_p$ decreases, i.e., the rate of oscillation increases.

**Remark.** Physically, $F$ is non-negative but to allow use of complex notation to aid in analysis, mathematically we allow $F \in (-\infty, \infty)$.

### 1.4.2 Discrete-time Case

Consider

$$x(n) = A\cos(\omega n + \theta),\ n \in \mathbf{Z}$$

- $A$ is the amplitude
- $\omega$ is the frequency (radians per sample)
- $n$ is the sample number
- $\theta$ is the phase (radians).

We also write $\omega = 2\pi f$, $f$ is frequency (cycles per sample).

If there exist integers $N$ such that

$$x(n+N) = x(n) \; \forall n$$

then the smallest such positive $N$ is the *fundamental period.* $x(n)$ is then said to be *periodic* of period $N$.

**Claim:** For a sinusoidal signal of the form

$$x(n) = A\cos(2\pi f_0 n + \theta)$$

then there exists an integer $N$ such that $x(n+N) = x(n) \; \forall n$ *if and only if* $f_0$ is a rational number.

***Proof*:** (only if part). Set

$$A\cos[2\pi f_0(n+N)+\theta] = A\cos(2\pi f_0 n + \theta).$$

This last equation is true $\Leftrightarrow \exists$ integer $k$ such that

$$2\pi f_0 N = 2k\pi$$

or,

$$f_0 = \frac{k}{N} \Rightarrow f_0 \text{ is rational.}$$

If you cancel common factors in $\frac{k}{N}$ so that the numerator and denominator are relatively prine, then the resulting denominator is the fundamental period of $x(n)$.

Consider

$$f = f_1 = \frac{1}{20} \Rightarrow N = 20$$

$$f = f_2 = \frac{1}{4} \Rightarrow N = 4$$

From the above we see that as $f$ increases the fundamental period decreases.

Again we allow negative $f$. Choose $\omega \in [-\pi, \pi] \Rightarrow f \in [-1/2, 1/2]$ ($\omega$ is chosen this way by convention since sinusoids are periodic of period $2\pi$). So,

$|f|_{max} = 1/2 \Rightarrow N = 2$ is the highest frequency in the discrete case.

## 1.5 Complex Exponentials

It is often convenient in analysis to represent signals or sequences using complex notation.

**Definition:** For any $\theta \in \mathbf{R}$,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

or,

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

**Note**: In this class $i = j = \sqrt{-1}$.

**Definition:** The *complex exponential sequence* is defined by

$$x(n) = e^{i\omega_0 n}$$

or,

$$x(n) = e^{i2\pi f_0 n}$$

where, $n \in \mathbf{Z}$, $\omega_0$ is the frequency (radians/sample) and $f_0$ is frequency (cycles/sample). $\omega_0$ is also called the *angular frequency*. The argument of $x(n)$ is $\arg x(n) = \omega_0 n$.

**Observations**

For $k \in \mathbf{Z}$,

$$e^{i(\theta+2\pi k)} = e^{i\theta}e^{i2\pi k} = (\cos\theta + i\sin\theta)(\cos 2\pi k + i\sin 2\pi k) = e^{i\theta}.$$

So, a complex number $e^{i\theta}$ does not change if its argument is changed by a multiple of $2\pi$. Thus, $x(n)$ will be periodic of period $N$, $N \in \mathbf{Z}^+$ (positive integers) *if and only if*

$$e^{i\omega_0 n} = e^{i\omega_0(n+N)} \Leftrightarrow \omega_0 N = 2\pi k$$

or

$$N = \frac{2\pi k}{\omega_0}.$$

This is equivalent to

$$N = \frac{2\pi k}{2\pi f_0}$$

or

$$f_0 = \frac{k}{N}$$

as before. So, if there exists $k \in \mathbf{Z}^+$ (use smallest $k$) such that $\frac{2\pi k}{\omega_0} \in \mathbf{Z}^+$ then $x(n) = e^{i\omega_0 n}$ has period $N = \frac{2\pi k}{\omega_0}$.

In the complex plane we have

$$z = x + iy,\ x \in \mathbf{R},\ y \in \mathbf{R},\ z \in \mathbf{C} \text{ (the complex numbers).}$$

$x(n) = e^{i\omega_0 n}$ then represents points on the unit circle in the complex plane.

Note:

$$|x(n)|^2 = x(n)x^*(n) = e^{i\omega_0 n}e^{-i\omega_0 n} = e^0 = 1.$$

***Example:*** Let $\omega_0 = \frac{\pi}{4}$ (45°). Find $N$, the fundamental period of $e^{i\omega_0 n}$.

$$N = \frac{2\pi k}{\omega_0} = \frac{2\pi k}{\pi/4} = 8k \text{ (take } k = 1) \Rightarrow N = 8.$$

So, $x(n) = \exp\left(i\frac{\pi}{4}n\right)$ has period 8.

***Example:*** Let $\omega_0 = \frac{3\pi}{4}$ (135°). Find $N$.

$$N = \frac{2\pi k}{3\pi/4} = \frac{8}{3}k \text{ (take } k = 3) \Rightarrow N = 8.$$