# EE 113 Midterm Solution

**Spring 2007**

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| Problem | Points | Score |
|---|---|---|
| 1 | 9 | |
| 2 | 11 | |
| 3 | 15 | |
| 4 | 15 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 10 | |
| 8 | 10 | |
| 9 | 10 | |
| | | |
| Total | 100 | |

**Problem 1.** For parts (a), (b) and (c) determine whether or not the system is

i. linear

ii. time-invariant

iii. BIBO stable, i.e., bounded input-bounded output stable

a. $y(n) = 2x(n)$.

**Solution.**

linear, time-invariant, BIBO stable

b. $y(n) = \log(n+1)x(n)$, log is the natural logarithm.

**Solution.**

linear, not time-invariant, not BIBO stable

c. $y(n) = [x(n)]^n$.

**Solution.**

not linear, not time-invariant, not BIBO stable

Note: You do not need to show any work on this problem if you can quickly recognize the answer.

**Problem 2.** Consider the system described by the following difference equation:

$$y(n) - \frac{7}{12}y(n-1) + \frac{1}{12}y(n-2) = x(n),$$

where,

$$x(n) = (1/2)^n\, u(n), \quad y(-1) = 1, \; y(-2) = 0.$$

a. Find the homogeneous solution for this system.

**Solution.**

$$\lambda^2 - \frac{7}{12}\lambda + \frac{1}{12} = \left(\lambda - \frac{1}{3}\right)\left(\lambda - \frac{1}{4}\right) = 0$$

so

$$y_h(n) = c_1\left(\frac{1}{3}\right)^n + c_2\left(\frac{1}{4}\right)^n.$$

b. Find the particular solution for this system.

**Solution.**

$$k\,(1/2)^n\,u(n) - \frac{7}{12}k\,(1/2)^{n-1}\,u(n-1) + \frac{1}{12}k\,(1/2)^{n-2}\,u(n-2) = (1/2)^n\,u(n)$$

or

$$ku(n) - \frac{14}{12}ku(n-1) + \frac{4}{12}ku(n-2) = u(n).$$

Evaluating this at $n = 2$ yields

$$k - \frac{14}{12}k + \frac{4}{12}k = 1$$

so $k = 6$. Thus,

$$y_p(n) = 6\,(1/2)^n\,, \quad n \geq 2.$$

c. Find the complete solution for this system.

**Solution.**

$$y(0) = \frac{7}{12}y(-1) - \frac{1}{12}y(-2) + x(0) = \frac{7}{12} + 1 = \frac{19}{12}$$

$$y(1) = \frac{7}{12}y(0) - \frac{1}{12}y(-1) + x(1) = \frac{133}{144} - \frac{1}{12} + \frac{1}{2} = \frac{193}{144}$$

$$y(n) = y_h(n) + y_p(n)$$

or

$$y(n) = c_1\left(\frac{1}{3}\right)^n + c_2\left(\frac{1}{4}\right)^n + 6\,(1/2)^n\,.$$

$$y(0) = c_1 + c_2 + 6 = \frac{19}{12}.$$

$$y(1) = \frac{c_1}{3} + \frac{c_2}{4} + 3 = \frac{193}{144}.$$

We find

$$c_1 = -6.67, \quad c_2 = 2.25.$$

$$y(n) = -6.67\left(\frac{1}{3}\right)^n + 2.25\left(\frac{1}{4}\right)^n + 6\,(1/2)^n\,, \quad n \geq 0.$$

d. Evaluate your $y(n)$ for $n = 0, 1, 2$.

**Solution.**

$$y(0) = 1.58, \quad y(1) = 1.34, \quad y(2) = 0.90.$$

**Problem 3.** Compute $X(z)$, the forward z-transform, (if it exists) for each of the following. Remember to specify the region of convergence in each case. If the forward z-transform does not exist, explain why.

a. $x(n) = \left(\frac{1}{2}\right)^n u(n-2)$.

**Solution.**

$$x(n) = \frac{1}{4}\left(\frac{1}{2}\right)^{n-2} u(n-2)$$

so

$$X(z) = \frac{1}{4}z^{-2}\frac{z}{z-\frac{1}{2}} = \frac{1}{4}\frac{z^{-1}}{z-\frac{1}{2}}, \quad |z| > \frac{1}{2}.$$

b. $x(n) = nu(n)$.

**Solution.**

$$X(z) = -z\frac{d}{dz}U(z) = -z\frac{d}{dz}\frac{z}{z-1} = \frac{z}{(z-1)^2}, \quad |z| > 1.$$

c.

$$x_1(n) = \begin{cases} \alpha^n u(n), & n \text{ is a multiple of } 2, \\ 0, & \text{elsewhere.} \end{cases}$$

$x(n) = x_1(2n)$.

**Solution.**

$$x(n) = \alpha^{2n} u(n) \Rightarrow X(z) = \frac{z}{z - \alpha^2}, \quad |z| > |\alpha^2|.$$

**Problem 4.** For parts (a) and (b) of the following compute $x(n)$, the inverse z-transform, using any method you wish. For part (c) use the residue method. Evaluate your expression for $x(n)$ at $n = 0, 1, 2$ in each case.

a. $X(z) = \dfrac{z^4 - 1}{z - \frac{1}{4}}$, ROC corresponds to a right-sided sequence.

**Solution.**

$$X(z) = z^3 \frac{z}{z - \frac{1}{4}} - z^{-1} \frac{z}{z - \frac{1}{4}}$$

so

$$x(n) = \left(\frac{1}{4}\right)^{n+3} u(n+3) - \left(\frac{1}{4}\right)^{n-1} u(n-1).$$

$$x(0) = \frac{1}{64}, \quad x(1) = -\frac{255}{256}, \quad x(2) = -\frac{255}{1024}.$$

b. $X(z) = \dfrac{z}{z^2 - 7z + 12}$, $\text{ROC} = \{z : |z| > 4\}$.

**Solution.**

$$X(z) = \frac{z}{(z-3)(z-4)} = \frac{-1}{z-3} + \frac{1}{z-4}$$

so

$$x(n) = -3^{n-1} u(n) + 4^{n-1} u(n-1).$$

$$x(0) = -\frac{1}{3}, \quad x(1) = 0, \quad x(2) = 1.$$

c. $X(z) = \dfrac{z}{(z-\frac{1}{2})(z-\frac{1}{4})}, \quad \text{ROC} = \left\{ z : \dfrac{1}{4} < |z| < \dfrac{1}{2} \right\}.$

**Solution.**

Since the numerator is of less degree than the denominator we have

$$x(n) = \sum_{\substack{\text{all poles}\\ \text{inside C}}} \text{Res } X(z)z^{n-1}, \quad m \geq 0$$
$$- \sum_{\substack{\text{all poles}\\ \text{outside C}}} \text{Res } X(z)z^{n-1}, \quad m < 0$$

where, $m$ is the least degree of the numerator polynomial of $X(z)z^{n-1}$. So

$$x(n) = \sum_{\substack{\text{all poles}\\ \text{inside C}}} \text{Res } \frac{z}{(z-\frac{1}{2})(z-\frac{1}{4})} z^{n-1}, \quad m \geq 0$$
$$- \sum_{\substack{\text{all poles}\\ \text{outside C}}} \text{Res } \frac{z}{(z-\frac{1}{2})(z-\frac{1}{4})} z^{n-1}, \quad m < 0$$

or

$$x(n) = \sum_{\substack{\text{all poles}\\ \text{inside C}}} \text{Res } \frac{z^n}{(z-\frac{1}{2})(z-\frac{1}{4})}, \quad m \geq 0$$
$$- \sum_{\substack{\text{all poles}\\ \text{outside C}}} \text{Res } \frac{z^n}{(z-\frac{1}{2})(z-\frac{1}{4})}, \quad m < 0.$$

In our case $m = n$ so

$$x(n) = \frac{z^n}{(z-\frac{1}{2})(z-\frac{1}{4})} \left( z - \frac{1}{4} \right) \Big|_{z=\frac{1}{4}} u(n)$$
$$- \frac{z^n}{(z-\frac{1}{2})(z-\frac{1}{4})} \left( z - \frac{1}{2} \right) \Big|_{z=\frac{1}{2}} u(n)$$

or

$$x(n) = -\left(\frac{1}{4}\right)^{n-1} u(n) - \left(\frac{1}{2}\right)^{n-2} u(-n-1).$$

$$x(0) = -4, \quad x(1) = -1, \quad x(2) = -\frac{1}{4}.$$

**Problem 5.** Evaluate the following infinite sum:

$$S = \sum_{n=0}^{\infty} n \left(\frac{1}{3}\right)^n.$$

**Solution.** Let us write $S$ as

$$S = \sum_{n=0}^{\infty} n 3^{-n}.$$

Let $x(n) = u(n)$ and let $x_1(n) = nx(n)$. Then,

$$X_1(z) = -z \frac{d}{dz} X(z).$$

$$X(z) = \mathcal{Z}\left[u(n)\right] = \frac{1}{1-z^{-1}}, \ |z| > 1,$$

$$\Rightarrow X_1(z) = \frac{z^{-1}}{(1-z^{-1})^2}.$$

We note that

$$S = X_1(z)\Big|_{z=3} \Rightarrow S = \frac{3}{4}.$$

**Problem 6.** Suppose $x(n)$ is a real ($x(n) = x^*(n)$) and even ($x(n) = x(-n)$) sequence with z-transform $X(z)$. Suppose $z_0$ is a zero of $X(z)$, i.e., $X(z_0) = 0$ for some complex number $z_0$.

a. Show $1/z_0$ is also a zero of $X(z)$.

**Solution.**

$$X(z) = \sum_n x(n)z^{-n}, \quad X(Z_0) = \sum_n x(n)z_0^{-n} = 0$$

so

$$X\left(z_0^{-1}\right) = \sum_n x(n)z_0^{n} = \sum_n x(-n)z_0^{-n} = \sum_n x(n)z_0^{-n} = 0$$

so $1/z_0$ is also a zero of $X(z)$.

b. Are there any other zeros of $X(z)$ implied by the information given? If so, find them.

**Solution.** Yes.

$$X^*(z) = \sum_n x^*(n)z^{*^{-n}} = \sum_n x(n)z^{*^{-n}} = X(z^*).$$

Now

$$X(z_0) = 0 \Rightarrow X^*(z_0) = 0^* = 0 \Rightarrow X(z_0^*) = 0$$

so $z_0^*$ is also a zero of $X(z)$. Also,

$$X^*\left(z_0^{-1}\right) = 0^* = 0 \text{ by part a so } X\left(z_0^{*^{-1}}\right) = 0$$

so $1/z_0^*$ is also a zero of $X(z)$.

**Problem 7.** Suppose $H(z)$, the z-transform of $h(n)$, is

$$H(z) = \frac{z}{z - \frac{1}{4}}, \quad |z| > \frac{1}{4}$$

and you are given the sequence

$$h_2(n) = \begin{cases} nh(n), & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

Find $H_2(z)$.

**Solution.**

Let

$$h_1(n) = \begin{cases} h(n), & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

Then $h_2(n) = nh_1(n)$. Now

$$H_1(z) = \sum_{n \ even} h(n)z^{-n} = \sum_n \frac{h(n) + (-1)^n h(n)}{2} z^{-n}$$

$$= \frac{1}{2}\sum_n h(n)z^{-n} + \frac{1}{2}\sum_n h(n)(-z)^{-n} = \frac{H(z) + H(-z)}{2}$$

$$= \frac{z/2}{z - \frac{1}{4}} + \frac{-z/2}{-z - \frac{1}{4}}, \quad |z| > \frac{1}{4}$$

$$H_2(z) = -z\frac{d}{dz}H_1(z) = \frac{z/8}{\left(z - \frac{1}{4}\right)^2} - \frac{z/8}{\left(-z - \frac{1}{4}\right)^2}$$

or

$$H_2(z) = \frac{z}{8} \cdot \frac{z}{\left(z - \frac{1}{4}\right)^2 \left(z + \frac{1}{4}\right)^2}, \quad |z| > \frac{1}{4}$$

**Problem 8.** A proposed form for a z-transform of a signal, $x(n)$, is given as

$$X(z) = \frac{z}{z-\alpha} + \frac{z}{z-\beta}$$

subject to the following constraints:

1. $|\alpha| \neq |\beta|$.
2. $\alpha + \beta = 3$.
3. $x(n)x(-n) = -2$ when $n = 1$.

Given the constraints, if their are any valid regions of convergence for $X(z)$, find them. If none exist, then show why not. You may assume without loss of generality that $|\alpha| > |\beta|$.

**Solution.** Try

i. $|z| > |\alpha|$. $x(n) = \alpha^n u(n) + \beta^n u(n)$. $x(1) = \alpha + \beta = 3$. $x(-1) = 0$. $x(1)x(-1) = 0$ so constraint 3 is violated.

ii. $|z| < |\beta|$. $x(n) = -\alpha^n u(-n-1) - \beta^n u(-n-1)$. $x(1) = 0$. $x(1)x(-1) = 0$ so constraint 3 is violated.

iii. $|\beta| < |z| < |\alpha|$. $x(n) = -\alpha^n u(-n-1) + \beta^n u(n)$.

$$x(1) = \beta, \quad x(-1) = -\frac{1}{\alpha}.$$

$$x(1)x(-1) = -\frac{\beta}{\alpha} = -2 \Leftrightarrow \beta = 2\alpha.$$

$$\alpha + \beta = 3 \Rightarrow \alpha + 2\alpha = 3 \Rightarrow \alpha = 1 \Rightarrow \beta = 2.$$

But, this contradicts $|\alpha| > |\beta|$ so no valid ROC exists.

**Problem 9.**

a. Consider the discrete-time signal

$$x(n) = n^{\log n} u(n-1)$$

where log denotes the natural logarithm. Determine if the z-transform of this signal exists and if it does exist find the region of convergence.

Note you do not have to actually find the z-transform. Justify your answer for credit.

**Solution.**

$$X(z) = \sum_{n=1}^{\infty} n^{\log n} z^{-n} = \sum_{n=1}^{\infty} \left( \frac{\left(n^{\log n}\right)^{1/n}}{z} \right)^n = \sum_{n=1}^{\infty} \left( \frac{n^{\frac{1}{n}\log n}}{z} \right)^n = \sum_{n=1}^{\infty} \left( \frac{n^{\log n^{1/n}}}{z} \right)^n.$$

At this point we will proceed with a heuristic argument instead of a rigorous argument since this would be acceptable on an exam. In order for the z-transform to exist we need the numerator in this last expression to be bounded. Using some large values of $n$ you can obsereve that

$$n^{1/n} \to 1 \text{ as } n \to \infty$$

so

$$\log n^{1/n} \to 0 \text{ as } n \to \infty$$

so

$$n^{\log n^{1/n}} \to 1 \text{ as } n \to \infty$$

and thus the z-transform does exist with ROC $= \{z : |z| > 1\}$.

b. Consider the discrete-time signal

$$x_N(n) = n^{\log n} u(n - N).$$

Find the function $X_N(z)$ that the z-transform of this signal approaches as $N$ approaches infinity

**Solution.**

$$X(z) = \sum_{n=N}^{\infty} n^{\log n} z^{-n} = \sum_{n=N}^{\infty} \left( \frac{n^{\log n^{1/n}}}{z} \right)^n.$$

As $N \to \infty$ we have

$$X(z) = \sum_{n=N}^{\infty} n^{\log n} z^{-n} \to \sum_{n=N}^{\infty} \left( \frac{1}{z} \right)^n = \sum_{n=N}^{\infty} z^{-n} = \frac{z^{-N}}{1 - z^{-1}}, \quad |z| > 1.$$

Thus,

$$X_N(z) = \frac{z^{-N}}{1 - z^{-1}}, \quad |z| > 1.$$