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EE 132A

Homework 5 Solutions

Problem 1. Proakis & Salehi 8.2 except replace $x(t)$ with

$$x(t) = \cos(\pi t/8)$$

Solution: 1) Noting that $\int \cos(\pi t/8) = \frac{8}{\pi}\sin(\pi t/8)$ we obtain:

$$\begin{aligned}
c_1 &= \int_0^4 x(t)\psi_1(t)dt = \frac{1}{2}\int_0^2 \cos(\pi t/8)dt - \frac{1}{2}\int_2^4 \cos(\pi t/8)dt = \frac{4}{\pi}\left[2\sin(\pi/4) - \sin(\pi/2)\right] \\
&= \frac{4}{\pi}\left[\sqrt{2} - 1\right] \\
c_2 &= \int_0^4 x(t)\psi_2(t)dt = \frac{4}{\pi}\sin(\pi/2) = \frac{4}{\pi} \\
c_3 &= \int_0^4 x(t)\psi_3(t)dt = \frac{1}{2}\int_0^1 \cos(\pi t/8)dt - \frac{1}{2}\int_1^2 \cos(\pi t/8)dt + \frac{1}{2}\int_2^3 \cos(\pi t/8)dt - \frac{3}{4}\int_1^2 \cos(\pi t/8)dt \\
&= \frac{4}{\pi}\left[2\sin(\pi/8) + 2\sin(3\pi/8) - \sqrt{2} - 1\right]
\end{aligned}$$

2)

$$E_{\text{min}} = \int_0^4 [x(t) - \hat{x}(t)]^2 dt = \int_0^4 x^2(t)dt - \sum_{i=1}^{3} c_i^2 = 2 - \sum_{i=1}^{3} c_i^2 = 0.0366$$

Problem 2. Proakis & Salehi 8.5 except change the pulse durations in Figure P-8.5 from (0 - 1), (2 - 3) to (0 - 2), (3 - 5).

Solution: 1) The impulse response of the matched filter is $h(t) = \psi(T - t) = \psi(5 - t)$, where $\psi(t) = \frac{1}{2A}s(t)$. A sketch of $h(t)$ is shown in Figure 1 (top).

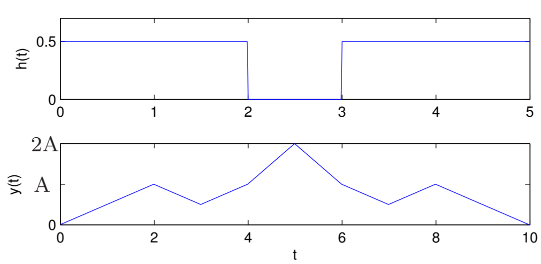

Figure 1: Figure of problem 2

2) The output of the matched filter is $y(t) = s(t) * h(t)$. A sketch of the output is shown in Figure 1 (bottom).

3) The output due to noise is $y_n = n(t) * h(t)|_{t=T}$. The output noise power is therefore

$$P_n = E[y_n^2] = \frac{N_0}{2}\int_{-\infty}^{\infty} h^2(t)dt = \frac{N_0}{2}$$

4) For a received signal $y = y_s + y_n$

$$P_e = P(y \le 0|y_s = 2A)P(y_s = 2A) + P(y > 0|y_s = -2A)P(y_s = -2A)$$

Assuming the symbols are equiprobable, and noting that $P(y \le 0|y_s = 2A) = P(y > 0|y_s = -2A)$ we obtain:

$$\begin{aligned} P_e &= P(y > 0|y_s = -2A) = P(-2A + y_n > 0) = P(y_n > 2A) = P(\tfrac{y_n}{\sqrt{N_0/2}} > \tfrac{2A}{\sqrt{N_0/2}}) \\ &= Q\left(\tfrac{2A}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\tfrac{8A^2}{N_0}}\right) \end{aligned}$$

Alternatively, since the average energy per bit is $E_b = 4A^2$, we obtain

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Problem 3. Proakis & Salehi 8.8 part (1) only.

Solution: 1) The output of the demodulator is

$$y(T) = \int_0^T (s_m(t) + n(t))dt = \sqrt{\frac{E_b}{T}}T + \int_0^T n(t)dt$$

Let $y_n = \int_0^T n(t)dt$. Then,

$$E[y_n^2] = \int_0^T\int_0^T n(t)n(\tau)dtd\tau = \frac{N_0}{2}\int_0^T\int_0^T \delta(t-\tau)dtd\tau = \frac{N_0T}{2}$$

The SNR at the output of the demodulator at $t = T$ is:

$$\left(\frac{S}{N}\right) = \frac{E_bT}{N_0T/2} = \frac{2E_b}{N_0}$$

Problem 4. Proakis & Salehi 8.10 except change R to $R = 10^6$ bits/sec and change P_2 to $P_2 = 10^{-5}$.

Solution: The probability of error for binary antipodal PAM is $P_e = Q(\sqrt{2E_b/N_0})$. To obtain $P_e = 10^{-5}$ we need $\sqrt{2E_b/N_0} = 4.2648$. The symbol rate is 10^6 symbols per second and therefore the symbol duration is $T = 10^{-6}$ seconds. Now, the average energy per bit is $E_b = A^2T$. Then we have $A = 301.57$.

Problem 5. Proakis & Salehi 8.13 except use

$$P(a_m = 1) = 1/4, \quad P(a_m = -1) = 3/4.$$

Solution: 1) The optimal threshold v can be obtained from:

$$P(a_m = 1) f_{Y|a_m=1}(v) = P(a_m = -1) f_{Y|a_m=-1}(v)$$

Since the noise is Gaussian, the random variable $y = a_m\sqrt{E_b} + y_n$ has conditional distribution:

$$f_{Y|a_m}(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-a_m\sqrt{E_b})^2}{2\sigma^2}}$$

Then we have:

$$\frac{1}{4}\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(v-\sqrt{E_b})^2}{2\sigma^2}} = \frac{3}{4}\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(v+\sqrt{E_b})^2}{2\sigma^2}}$$

and

$$(v+\sqrt{E_b})^2 - (v-\sqrt{E_b})^2 = 2\sigma^2 \ln 3 \quad \Rightarrow \quad v = \frac{\sigma^2 \ln 3}{2\sqrt{E_b}} = \frac{N_0 \ln 3}{4\sqrt{E_b}}$$

2) The probability of error is given by:

$$P_e = \frac{1}{4}P(y \le v | a_m = 1) + \frac{3}{4}P(y > v | a_m = -1) = \frac{1}{4}P(\sqrt{E_b} + y_n \le v) + \frac{3}{4}P(-\sqrt{E_b} + y_n > v)$$

where $y_n \sim \mathcal{N}(0, N_0/2)$. Thus,

$$\begin{aligned} P_e &= \tfrac{1}{4}P\left(\tfrac{y_n}{\sqrt{N_0/2}} \le \tfrac{v-\sqrt{E_b}}{\sqrt{N_0/2}}\right) + \tfrac{3}{4}P\left(\tfrac{y_n}{\sqrt{N_0/2}} > \tfrac{v+\sqrt{E_b}}{\sqrt{N_0/2}}\right) = \tfrac{1}{4}Q\left(\tfrac{\sqrt{E_b}-v}{\sqrt{N_0/2}}\right) + \tfrac{3}{4}Q\left(\tfrac{\sqrt{E_b}+v}{\sqrt{N_0/2}}\right) \\ &= \tfrac{1}{4}Q\left(\sqrt{\tfrac{2E_b}{N_0}} - \sqrt{\tfrac{N_0}{2E_b}}\tfrac{\ln 3}{2}\right) + \tfrac{3}{4}Q\left(\sqrt{\tfrac{2E_b}{N_0}} + \sqrt{\tfrac{N_0}{2E_b}}\tfrac{\ln 3}{2}\right) \end{aligned}$$

3) Using $E_b = 1$ and $N_0 = 0.1$ we obtain:

$$v = 0.0275$$

and

$$P_e = 3.3302 \times 10^{-6}$$

Problem 6. Proakis & Salehi 8.20.

Solution: 1) Assuming the symbols are equiprobable, the average energy per bit is:

$$E_b = \frac{1}{2} \times A^2 T + \frac{1}{2} \times 0 = \frac{A^2 T}{2}$$

The symbols $s_0(t)$ and $s_1(t)$ can be represented as:

$$s_m(t) = a_m A\sqrt{T}\psi(t) = a_m\sqrt{2}\sqrt{E_b}\psi(t) \qquad m = 1, 2$$

where $a_0 = 0$, $a_1 = 1$, and $\psi(t)$ is a unitary energy rectangular pulse of duration T and centered at $T/2$ (the amplitude is therefore $\sqrt{1/T}$). The matched filter can be found to be $h(t) = \psi(T-t) = \psi(t)$. The signal at time $t = T$ at the output of the matched filter will be

$y = y_s + y_n$, where $y_s = a_m\sqrt{2}\sqrt{E_b}$ and y_n is zero-mean Gaussian noise with variance $N_0/2$. Since the symbols are equiprobable, the optimal threshold can be found by solving:

$$e^{-\frac{(v-\sqrt{2E_b})^2}{2\sigma^2}} = e^{-\frac{v^2}{2\sigma^2}}$$

and we obtain for the optimum threshold:

$$v = \sqrt{E_b/2} = \frac{A\sqrt{T}}{2}$$

The optimal detector is:

$$y \underset{s_1}{\overset{s_0}{\lesseqgtr}} \sqrt{E_b/2}$$

2) The probability of error is

$$P_e = P(y_n > v) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

For antipodal signals we had

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Therefore on-off keying requires twice the energy of antipodal signalling to obtain the same probability of bit error.

Problem 7. Proakis & Salehi 8.23 except in part (2) use an error probability of 10^{-6}.

Solution: 1) Since the symbols are equiprobable and the Laplacian pdf is symmetric, the optimal threshold is set at 0. The probability of error is:

$$P_e = \frac{1}{2}P(A+n \le 0) + \frac{1}{2}P(-A+n > 0) = P(-A+n > 0) = P(n > A)$$

$$P_e = \int_A^\infty \frac{1}{\sqrt{2\sigma^2}} e^{-n\sqrt{2}/\sigma} dn = \frac{1}{2} e^{-\frac{\sqrt{2}A}{\sigma}}$$

2) For $P_e = 10^{-6}$ we need $A/\sigma = 9.2789$. The signal-to-noise ratio is

$$\left(\frac{S}{N}\right) = \frac{A^2}{\sigma^2} = 86.1 = 19.35 \text{ dB}$$

For Gaussian noise, to achieve $P_e = 10^{-6} = Q(\sqrt{(S/N)})$ we need $\sqrt{(S/N)} = 4.7534$. Therefore,

$$\left(\frac{S}{N}\right) = 22.59 = 13.54 \text{ dB}$$

Thus for Laplacian noise we need 5.8 dB more SNR to achieve the same performance as in the Gaussian case.