EE 132A

Homework 3 Solutions

Problem 1. Proakis & Salehi 2.16 parts 2 and 13.

Solution: 2) Nolinear, if we multiply the input by a constant -1, the output does not change. In a linear system the output should be scaled by -1.

13) Linear, we can write the output of this feedback system as

$$y(t) = x(t) + y(t-1) = \sum_{n=0}^{\infty} x(t-n)$$

Then for $x(t) = \alpha x_1(t) = \beta x_2(t)$

$$y(t) = \sum_{n=0}^{\infty} [\alpha x_1(t-n) + \beta x_2(t-n)] = \alpha \sum_{n=0}^{\infty} x_1(t-n) + \beta \sum_{n=0}^{\infty} x_2(t-n) = \alpha y_1(t) + \beta y_2(t)$$

Problem 2. Proakis & Salehi 2.24 parts 1 and 9.

Solution: 1) The invariant: The response to $x(t-t_0)$ is $2x(t-t_0)+3$ which is $y(t-t_0)$.

9) Time-invariant system. Writing $y(t)$ as $\sum_{n=0}^{\infty} x(t-n)$ we get

$$y(t-t_0) = \sum_{n=0}^{\infty} x(t-t_0-n) = T[x(t-t_0)]$$

Problem 3. Proakis & Salehi 5.6.

Solution: 1) X can take four different values: 0 if no head shows up, 1 if only one head shows up in the four flips of the coin, 2 for two heads, and 3 if the outcome of each flip is head.

2) X follows the binomial distribution with $n = 3$. Thus

$$P(X = k) = \begin{cases} \begin{pmatrix} 3 \\ k \end{pmatrix} p^k (1-p)^{3-k} & \text{for } 0 \leq k \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

3)

$$F_X(k) = \sum_{m=0}^{k} \begin{pmatrix} 3 \\ m \end{pmatrix} p^m (1-p)^{3-m}$$

Hence

$$F_X(k) = \begin{cases} 0 & k < 0 \\ (1-p)^3 & k = 0 \\ (1-p)^3 + 3p(1-p)^2 & k = 1 \\ (1-p)^3 + 3p(1-p)^2 + 3p^2(1-p) & k = 2 \\ (1-p)^3 + 3p(1-p)^2 + 3p^2(1-p) + p^3 = 1 & k = 3 \\ 1 & k > 3 \end{cases}$$

A plot of $F_X(k)$ is shown in Figure 1.

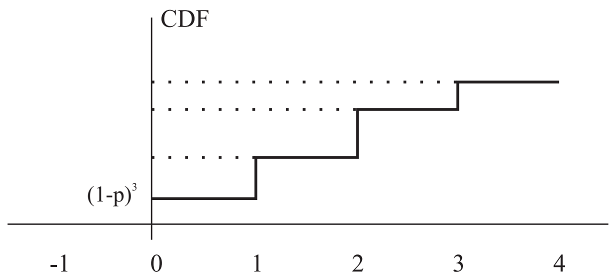

Figure 1: Figure of problem 3 (5.6.3)

4)

$$P(X > 1) = \sum_{k=2}^{3} \binom{3}{k} p^k (1-p)^{3-k} = 3p^2(1-p) + (1-p)^3$$

Problem 4. Proakis & Salehi 5.10.

Solution: In general, if X is a Gaussian RV with mean m and variance σ^2, we have,

$$P(X > \alpha) = Q\left(\frac{\alpha - m}{\sigma}\right)$$

Therefore,

$$P(X > 7) = Q\left(\frac{7-4}{3}\right) = Q(1) = 0.158$$

and using the relation $P(0 < X < 9) = P(X > 0) - (X > 9)$ we have

$$P(0 < X < 9) = Q\left(\frac{0-4}{3}\right) - Q\left(\frac{9-4}{3}\right) = 1 - Q(1.33) - Q(1.66) \approx 0.858$$

Problem 5. Proakis & Salehi 5.38.

Solution: 1)

$$\begin{aligned} m_x(t) &= E[X(t)] = E[X\cos(2\pi f_0 t) + E[Y\sin(2\pi f_0 t)] \\ &= E[X]\cos(2\pi f_0 t) + E[Y]\sin(2\pi f_0 t) \\ &= 0 \end{aligned}$$

where the last equality follows from the fact that $E[X] = E[Y] = 0$.

2)

$$\begin{aligned}
R_x(t+\tau,t) &= E\left\{[X\cos(2\pi f_0(t+\tau)) + Y\sin(2\pi f_0(t+\tau))][X\cos(2\pi f_0 t) + Y\sin(2\pi f_0 t)]\right\} \\
&= E[X^2\cos(2\pi f_0(t+\tau))\cos(2\pi f_0 t) + E[XY\cos(2\pi f_0(t+\tau))\sin(2\pi f_0 t)] + \\
&\quad E[YX\sin(2\pi f_0(t+\tau))\cos(2\pi f_0 t)] + E[Y^2\sin(2\pi f_0(t+\tau))\sin(2\pi f_0 t)] \\
&= \frac{\sigma^2}{2}[\cos(2\pi f_0(2t+\tau)) + \cos(2\pi f_0\tau)] + \frac{\sigma^2}{2}[\cos(2\pi f_0\tau) - \cos(2\pi f_0(2t-\tau))] \\
&= \sigma^2\cos(2\pi f_0\tau)
\end{aligned}$$

where we have used the fact that $E[XY] = 0$. Thus, the process is stationary for $R_x(t+\tau,t)$ depends only on τ.

3) The power spectral density is the Fourier transform of the autocorrelation function, hence

$$S_x(f) = \frac{\sigma^2}{2}[\delta(f-f_0) + \delta(f+f_0)].$$

4) If $\sigma_x^2 \neq \sigma_y^2$, then

$$m_x(t) = E[X]\cos(2\pi f_0 t) + E[Y]\sin(2\pi f_0 t) = 0$$

and

$$\begin{aligned}
R_x(t+\tau,t) &= E[X^2]\cos(2\pi f_0(t+\tau))\cos(2\pi f_0 t) + E[Y^2]\sin(2\pi f_0(t+\tau))\sin(2\pi f_0 t) \\
&= \frac{\sigma_x^2}{2}[\cos(2\pi f_0(2t+\tau)) - \cos(2\pi f_0\tau)] + \frac{\sigma_y^2}{2}[\cos(2\pi f_0\tau) - \cos(2\pi f_0(2t+\tau))] \\
&= \frac{\sigma_x^2 - \sigma_y^2}{2}\cos(2\pi f_0(2t+\tau) + \frac{\sigma_x^2 + \sigma_y^2}{2}\cos(2\pi f_0\tau)
\end{aligned}$$

The process is not stationary for $R_x(t+\tau,t)$ does not depend only on τ but on t as well. However the process is cyclostationary with period $T_0 = \frac{1}{2f_0}$. Note that if X or Y is not of zero mean then the period of the cyclostationary process is $T_0 = \frac{1}{f_0}$.

Problem 6. Proakis & Salehi 5.40.

Solution: 1) $S_x(f) = \frac{N_0}{2}, R_x(\tau) = \frac{N_0}{2}\delta(\tau)$. The autocorrelation function and the power spectral density of the output are given by

$$R_Y(t) = R_x(\tau) \star h(\tau) \star h(-\tau), \qquad S_Y(f) = S_X(f)|H(f)|^2$$

With $H(f) = \Pi(f/2B)$ we have $|H(f)|^2 = \Pi^2(f/2B) = \Pi(f/2B)$ so that

$$S_Y(f) = \frac{N_0}{2}\Pi(\frac{f}{2B})$$

Taking the inverse Fourier transform of the previous we obtain the autocorrelation function of the output

$$R_Y(\tau) = BN_0\text{sinc}(2B\tau)$$

2) The output random process $Y(t)$ is a zero-mean Gaussian process with variance

$$\sigma^2_{Y(t)} = \mathrm{E}[Y^2(t)] = \mathrm{E}[Y^2(t+\tau)] = R_Y(0) = BN_0$$

The correlation coefficient of the jointly Gaussian process $Y(t+\tau), Y(t)$ is

$$\rho_{Y(t+\tau)Y(t)} = \frac{\mathrm{COV}(Y(t+\tau)Y(t))}{\sigma_{Y(t+\tau)}\sigma_{Y(t)}} = \frac{\mathrm{E}[Y(t+\tau)Y(t)]}{BN_0} = \frac{R_Y(\tau)}{BN_0}$$

With $\tau = \frac{1}{2B}$ we have $R_Y(1/2B) = 0$ so that $\rho_{Y(t+\tau)Y(t)} = 0$. Hence, the joint probability density function of $Y(t)$ and $Y(t+\tau)$ is

$$f_{Y(t+\tau),Y(t)} = \frac{1}{2\pi BN_0} e^{-\frac{Y^2(t+\tau)+Y^2(t)}{2BN_0}}$$

Since the processes are Gaussian and uncorrelated, they are also independent.

Problem 7. Proakis & Salehi 5.58.

Solution: 1) The power spectral density $S_n(f)$ is depicted in Figure 2. The output bandpass process has non-zero power content for frequencies in the band $49 \times 10^6 \leq |f| \leq 51 \times 10^6$. The power content is

$$P = \int_{-51\times10^6}^{-49\times10^6} 10^{-8}\left(1 + 10^{-8} f\right) df + \int_{49\times10^6}^{51\times10^6} 10^{-8}\left(1 - 10^{-8} f\right) df = 2 \times 10^{-2}$$

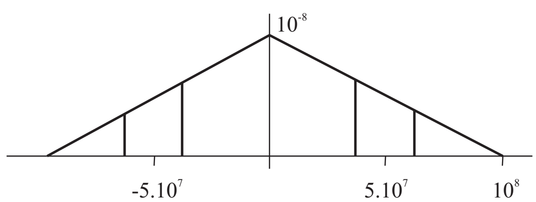

Figure 2: Figure of problem 7 (5.58.1)

2) The output process $N(t)$ can be written as

$$N(t) = N_c(t)\cos(2\pi 50 \times 10^6 t) - N_s(t)\sin(2\pi 50 \times 10^6 t)$$

where $N_c(t)$ and $N_s(t)$ are the in-phase and quadrature components respectively, given by

$$\begin{aligned} N_c(t) &= N(t)\cos(2\pi 50 \times 10^6 t) + \hat{N}(t)\sin(2\pi 50 \times 10^6 t) \\ N_s(t) &= \hat{N}(t)\cos(2\pi 50 \times 10^6 t) - N(t)\sin(2\pi 50 \times 10^6 t) \end{aligned}$$

The power content of the in-phase component is given by

$$\begin{aligned} E[|N_c(t)|^2] &= E[|N(t)|^2]\cos^2(2\pi 50 \times 10^6 t) + E[|\hat{N}(t)|^2]\sin^2(2\pi 50 \times 10^6 t) \\ &= E[|N(t)|^2] = 2 \times 10^{-2} \end{aligned}$$

where we have used the fact that $E[|N(t)|^2] = E[|\hat{N}(t)|^2]$. Similarly we find that $E[|N_s(t)|^2] = 2 \times 10^{-2}$.

3) The power spectral density of $N_c(t)$ and $N_s(t)$ is

$$S_{N_c}(f) = S_{N_s}(f) = \begin{cases} S_N(f - 50 \times 10^6) + S_N(f + 50 \times 10^6) & |f| \leq 50 \times 10^6 \\ 0 & \text{otherwise} \end{cases}$$

$S_{N_c}(f)$ is depicted in Figure 3. The power content of $N_c(t)$ can now be found easily by

$$P_{N_c} = P_{N_s} = \int_{-10^6}^{10^6} 10^{-8} df = 2 \times 10^{-2}$$

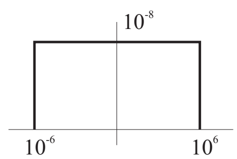

Figure 3: Figure of problem 7 (5.58.3)

4) The power spectral density of the output is given by

$$S_Y(f) = S_X(f)|H(f)|^2 = (|f| - 49 \times 10^6)(10^{-8} - 10^{-16}|f|) \quad \text{for } 49 \times 10^6 \leq |f| \leq 51 \times 10^6$$

Hence, the power content of the output is

$$\begin{aligned} P_Y &= \int_{-51\times10^6}^{-49\times10^6} (-f - 49 \times 10^6)(10^{-8} + 10^{-16} f) df + \int_{49\times10^6}^{51\times10^6} (f - 49 \times 10^6)(10^{-8} - 10^{-16} f) df \\ &= 2 \times 10^4 - \frac{4}{3} 10^2 \end{aligned}$$

The power spectral density of the in-phase and quadrature components of the output process is given by

$$\begin{aligned} S_{Y_c}(f) = S_{Y_s}(f) &= ((f + 50 \times 10^6) - 49 \times 10^6)(10^{-8} - 10^{-16}(f + 50 \times 10^6)) \\ &+ (-(f - 50 \times 10^6) - 49 \times 10^6)(10^{-8} + 10^{-16}(f - 50 \times 10^6)) \\ &= 2 \times 10^{-16} f^2 + 10^{-2} \end{aligned}$$

for $|f| \leq 10^6$ and zero otherwise. The power content of the in-phase and quadrature component is

$$\begin{aligned} P_{Y_c} = P_{Y_s} &= \int_{-10^6}^{10^6} (-2 \times 10^{-16} f^2 + 10^{-2}) df = -2 \times 10^{-16} \frac{1}{3} f^3 \Big|_{-10^6}^{10^6} + 10^{-2} f \Big|_{-10^6}^{10^6} \\ &= 2 \times 10^4 - \frac{4}{3} 10^2 = P_Y \end{aligned}$$