EE 132A
Homework 1 Solutions

Problem 1. For $a \neq 0$ prove the scaling property of the Fourier transform:

$$x(at) \longleftrightarrow \frac{1}{|a|}X(f/a).$$

Solution: From the definition of the Fourier transform:

$$\mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(at)e^{-j2\pi ft}dt$$

Using the change of variables $u = at$, if a is positive, we obtain:

$$\mathcal{F}[x(at)] = \frac{1}{a}\int_{-\infty}^{\infty} x(u)e^{-j2\pi fu/a}du = \frac{1}{a}X(f/a)$$

If a is negative, we get:

$$\mathcal{F}[x(at)] = -\frac{1}{a}\int_{-\infty}^{\infty} x(u)e^{-j2\pi fu/a}du = -\frac{1}{a}X(f/a)$$

From these two results we conclude that

$$\mathcal{F}[x(at)] = \frac{1}{|a|}X(f/a)$$

Problem 2. Prove the time-shift property of the Fourier transform:

$$x(t - t_0) \longleftrightarrow e^{-j2\pi ft_0}X(f).$$

Solution: From the definition of the Fourier transform:

$$\mathcal{F}[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0)e^{-j2\pi ft}dt$$

Using the change of variables $u = t - t_0$ we obtain:

$$\begin{aligned}
\mathcal{F}[x(t-t_0)] &= \int_{-\infty}^{\infty} x(u)e^{-j2\pi f(u+t_0)}dt \\
&= e^{-j2\pi ft_0}\int_{-\infty}^{\infty} x(u)e^{-j2\pi fu}du \\
&= e^{-j2\pi ft_0}X(f)
\end{aligned}$$

Problem 3.

a. Compute the Fourier transform of the rectangular pulse, $x(t) = \prod(t)$, and sketch a plot of its graph in the frequency domain. You should produce two plots (one for the magnitude of the Fourier transform and one for the phase). In the magnitude plot, clearly identify where the maximum height occurs and the value of the maximum height. Also, identify where the zero-crossings occur.

b. Repeat part a for $x(t) = \prod(t)\cos(2\pi f_0 t)$ where $f_0 = 1$ kHz.

Solution: a) From the definition of the Fourier transform:

$$\mathcal{F}[\Pi(t)] = \int_{-\infty}^{\infty} \Pi(t)e^{-j2\pi ft}dt = \int_{-1/2}^{1/2} e^{-j2\pi ft}dt$$

When $f = 0$, it is evident that the result of the above integration becomes 1. When $f \neq 0$, we obtain:

$$\mathcal{F}[\Pi(t)] = \frac{e^{-j\pi f} - e^{j\pi f}}{-j2\pi f}$$

Using Euler's formula $\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{j2}$ we arrive at:

$$\mathcal{F}[\Pi(t)] = \left\{ \begin{array}{ll} \frac{\sin(\pi f)}{\pi f} & , f \neq 0 \\ 1 & , f = 0 \end{array} \right\} = \mathrm{sinc}(f)$$

The magnitude and phase plots are shown in Figure 1.

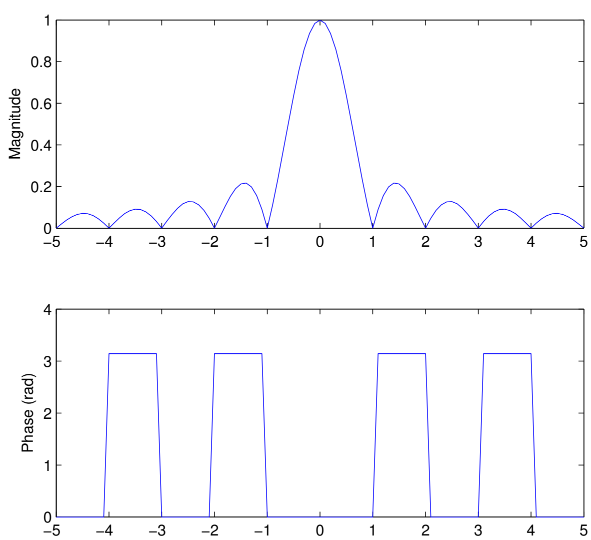

Figure 1: Figure of problem 3, part a

b) Using Euler's rule we can decompose the cosine as follows

$$x(t) = \Pi(t)\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} = \frac{1}{2}\Pi(t)e^{j2\pi f_0 t} + \frac{1}{2}\Pi(t)e^{-j2\pi f_0 t}$$

From the modulation property of the Fourier transform and using linearity, we obtain

$$X(f) = \frac{1}{2}\mathcal{F}[\Pi(t)]\Big|_{f-f_0} + \frac{1}{2}\mathcal{F}[\Pi(t)]\Big|_{f+f_0} = \frac{1}{2}\text{sinc}(f - f_0) + \frac{1}{2}\text{sinc}(f + f_0)$$

The magnitude and phase plots are shown in Figure 2.

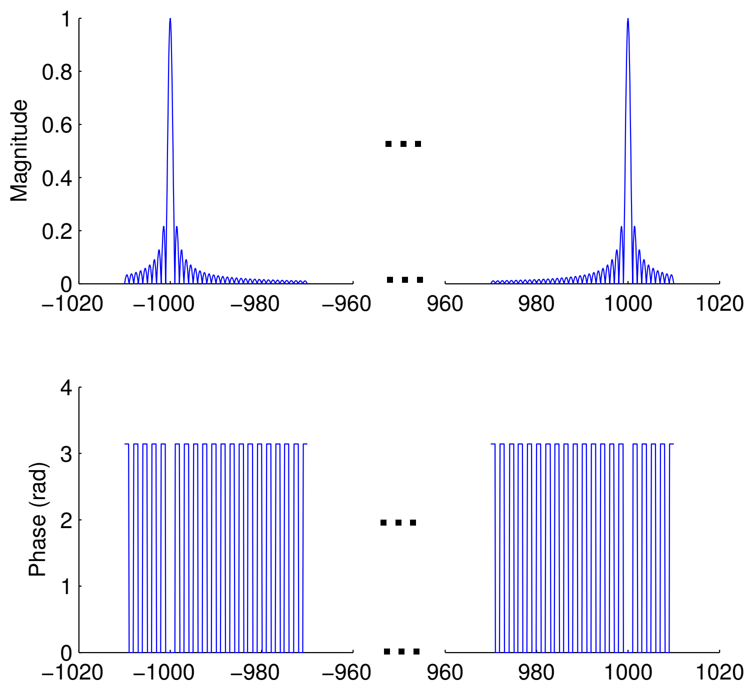

Figure 2: Figure of problem 3, part b

Problem 4. Proakis & Salehi 2.1 parts 1, 5, 7.

Solution: The plots are shown in Figure 3.

Problem 5. Proakis & Salehi 2.39 part 5.

Solution: $x_5(t) = 1$. It follows then that $x_{5,0} = 1$ and $x_{5,n} = 0$, $\forall n \neq 0$.

Problem 6. Using properties of the Fourier transform compute

$$\int_{-\infty}^{\infty} \left(\frac{\sin \pi f}{\pi f}\right)^2 df.$$

Solution: From the convolution property of the Fourier transform and the fact that $\mathcal{F}[\Pi(t)] = \text{sinc}(f)$ we know that

$$\mathcal{F}[\Pi(t) \star \Pi(t)] = \text{sinc}^2(f)$$

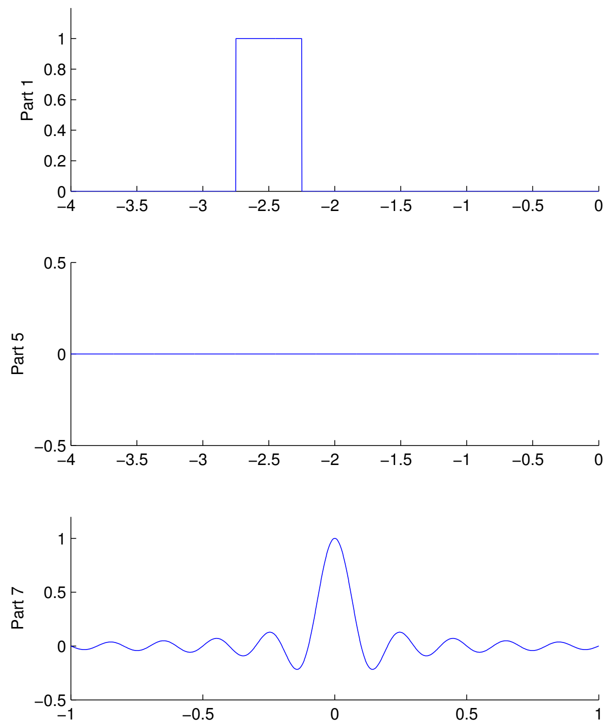

Figure 3: Figure of problem 4

Thus, from the definition of the inverse Fourier transform we obtain

$$\Pi(t) \star \Pi(t) = \int_{-\infty}^{\infty} \text{sinc}^2(f) e^{j2\pi f t} df$$

Evaluating at time $t = 0$, the above equation becomes

$$\Pi(t) \star \Pi(t)\Big|_{t=0} = \int_{-\infty}^{\infty} \text{sinc}^2(f) df$$

From the definition of convolution,

$$\Pi(t) \star \Pi(t)\Big|_{t=0} = \int_{-\infty}^{\infty} \Pi(\tau)\Pi(-\tau) d\tau = 1$$

so finally, we obtain

$$\int_{-\infty}^{\infty} \text{sinc}^2(f) df = \int_{-\infty}^{\infty} \left(\frac{\sin \pi f}{\pi f}\right)^2 df = 1$$

Problem 7. Let $\alpha > 0$, and consider the convolution

$$x(t) = e^{\alpha t} u(-t) * e^{-\alpha t} u(t)$$

where $u(t)$ is the unit-step function.

a. Compute $x(t)$ via direct convolution.

b. Compute $x(t)$ using a Fourier transform property.

Solution:
a)

$$x(t) = \int_{-\infty}^{\infty} e^{\alpha\tau}u(-\tau)e^{-\alpha(t-\tau)}u(t-\tau)d\tau = e^{-\alpha t}\int_{-\infty}^{0} e^{2\alpha\tau}u(t-\tau)d\tau$$

If $t > 0$, then $t - \tau$ will be positive for every negative value of τ, and we obtain

$$x(t) = e^{-\alpha t}\int_{-\infty}^{0} e^{2\alpha\tau}d\tau = \frac{e^{-\alpha t}}{2\alpha}u(t)$$

If $t < 0$, then $t - \tau$ will be positive in the interval $(-\infty, t)$ and we get

$$x(t) = e^{-\alpha t}\int_{-\infty}^{t} e^{2\alpha\tau}d\tau = e^{-\alpha t}\frac{e^{2\alpha t}}{2\alpha} = \frac{e^{\alpha t}}{2\alpha}u(-t)$$

Combining these two results we arrive finally at

$$x(t) = \frac{1}{2\alpha}\left[e^{-\alpha t}u(t) + e^{\alpha t}u(-t)\right] = \frac{e^{-\alpha|t|}}{2\alpha}$$

b) Now we solve the same problem by computing the Fourier transforms

$$\mathcal{F}[e^{-\alpha t}u(t)] = \frac{1}{\alpha + j2\pi f} \qquad \mathcal{F}[e^{\alpha t}u(-t)] = \frac{1}{\alpha - j2\pi f}$$

$$X(f) = \frac{1}{\alpha + j2\pi f}\frac{1}{\alpha - j2\pi f} = \frac{1}{2\alpha}\left[\frac{1}{\alpha + j2\pi f} + \frac{1}{\alpha - j2\pi f}\right]$$

So finally we obtain

$$x(t) = \frac{1}{2\alpha}\left[e^{-\alpha t}u(t) + e^{\alpha t}u(-t)\right] = \frac{e^{-\alpha|t|}}{2\alpha}$$

Problem 8. Proakis & Salehi 2.52.

Solution: 1) Clearly

$$x_1(t + kT_0) = \sum_{n=-\infty}^{\infty} x(t + kT_0 - nT_0) = \sum_{n=-\infty}^{\infty} x(t - (n-k)T_0) = \sum_{m=-\infty}^{\infty} x(t - mT_0) = x_1(t)$$

2) Noticing that $x(t) * \delta(t - nT_0) = x(t - nT_0)$ we have

$$x_1(t) = x(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

3) From the convolution property, we obtain

$$X_1(f) = X(f) \cdot \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T_0}) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} X(\frac{n}{T_0})\delta(f - \frac{n}{T_0})$$

Problem 9. Proakis & Salehi 2.53.

Solution: 1) By Parseval's theorem

$$\int_{-\infty}^{\infty} \text{sinc}^5(t)dt = \int_{-\infty}^{\infty} \text{sinc}^3(t)\text{sinc}^2(t)dt = \int_{-\infty}^{\infty} \Lambda(f)T(f)df$$

where

$$T(f) = \mathcal{F}[\text{sinc}^3(t)] = \Pi(t) * \Lambda(t)$$

But

$$\Pi(t) * \Lambda(t) = \int_{-\infty}^{\infty} \Pi(\theta)\Lambda(f-\theta)d\theta = \int_{-1/2}^{1/2} \Lambda(f-\theta)d\theta = \int_{f-1/2}^{f+1/2} \Lambda(v)dv$$

Now

$$T(f) = \begin{cases} 0 & \text{for } f \leq -3/2 \text{ or } f > 3/2 \\ \int_{-1}^{f+1/2}(v+1)dv = \frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8} & \text{for } -3/2 \leq f \leq -1/2 \\ \int_{f-1/2}^{0}(v+1)dv + \int_{0}^{f+1/2}(-v+1)dv = -f^2 + \frac{3}{4} & \text{for } -1/2 \leq f \leq 1/2 \\ \int_{f-1/2}^{1}(-v+1)dv = \frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8} & \text{for } 1/2 \leq f \leq 3/2 \end{cases}$$

Hence,

$$\begin{aligned} \int_{-\infty}^{\infty} \Lambda(f)T(f)df &= \int_{-1}^{-1/2}(\frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8})(f+1)df + \int_{-1/2}^{0}(-f^2 + \frac{3}{4})(f+1)df \\ &\quad + \int_{0}^{1/2}(-f^2 + \frac{3}{4})(-f+1)df + \int_{1/2}^{1}(\frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8})(-f+1)df \\ &= \frac{41}{64} \end{aligned}$$

2)

$$\begin{aligned} \int_{0}^{\infty} e^{-\alpha t}\text{sinc}(t)dt &= \int_{-\infty}^{\infty} e^{-\alpha t}u(t)\text{sinc}(t)dt \\ &= \int_{-\infty}^{\infty} \frac{1}{\alpha + j2\pi f}\Pi(f)df = \int_{-1/2}^{1/2} \frac{1}{\alpha + j2\pi f}df \\ &= \frac{1}{j2\pi}\ln(\alpha + j2\pi f)\Big|_{-1/2}^{1/2} = \frac{1}{j2\pi}\ln\left(\frac{\alpha + j\pi}{\alpha - j\pi}\right) = \frac{1}{\pi}\tan^{-1}\frac{\pi}{\alpha} \end{aligned}$$

3)

$$\begin{aligned} \int_{0}^{\infty} e^{-\alpha t}\text{sinc}^2(t)dt &= \int_{-\infty}^{\infty} e^{-\alpha t}u(t)\text{sinc}^2(t)dt \\ &= \int_{-\infty}^{\infty} \frac{1}{\alpha + j2\pi f}\Lambda(f)df = \int_{-1}^{0} \frac{f+1}{\alpha + j2\pi f} + \int_{0}^{1} \frac{-f+1}{\alpha + j2\pi f}df \end{aligned}$$

But $\int \frac{x}{a+bx}dx = \frac{x}{b} - \frac{a}{b^2}\ln(a+bx)$ so that

$$\begin{aligned}
\int_0^\infty e^{-\alpha t}\text{sinc}^2(t)dt &= \left(\frac{f}{j2\pi} + \frac{\alpha}{4\pi^2}\ln(\alpha + j2\pi f)\right)\Big|_{-1}^{0} \\
&= -\left(\frac{f}{j2\pi} + \frac{\alpha}{4\pi^2}\ln(\alpha + j2\pi f)\right)\Big|_{0}^{1} + \frac{1}{j2\pi}\ln(\alpha + j2\pi f)\Big|_{-1}^{1} \\
&= \frac{1}{\pi}\tan^{-1}(\frac{2\pi}{\alpha}) + \frac{\alpha}{2\pi^2}\ln(\frac{\alpha}{\sqrt{\alpha^2+4\pi^2}})
\end{aligned}$$

4)

$$\begin{aligned}
\int_0^\infty e^{-\alpha t}\cos(\beta t)dt &= \int_{-\infty}^{\infty} e^{-\alpha t}u(t)\cos(\beta t)dt \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{\alpha + j2\pi f}(\delta(f - \frac{\beta}{2\pi}) + \delta(f + \frac{\beta}{2\pi})df \\
&= \frac{1}{2}\left[\frac{1}{\alpha + j\beta} + \frac{1}{\alpha - j\beta}\right] = \frac{\alpha}{\alpha^2 + \beta^2}
\end{aligned}$$

Problem 10. Proakis & Salehi 2.7.

Solution: For the first two parts we need the integral $I = \int e^{-\alpha t}\cos^2(t)dt$. Using the relation $\cos^2(a) = \frac{1+cos(2a)}{2}$ we obtain

$$\begin{aligned}
I &= \int e^{-\alpha t}\cos^2(t)dt = \frac{1}{2}\int e^{-\alpha t}(1+\cos(2t))dt = -\frac{e^{-\alpha t}}{2\alpha} + \frac{1}{4}\int\left(e^{-(\alpha-j2)t} + e^{-(\alpha+j2)t}\right)dt \\
&= -\frac{e^{-\alpha t}}{2\alpha} - \frac{e^{-(\alpha-j2)t}}{4(\alpha-j2)} - \frac{e^{-(\alpha+j2)t}}{4(\alpha+j2)} = -e^{-\alpha t}\left[\frac{1}{2\alpha} + \frac{e^{j2t}}{4(\alpha-j2)} + \frac{e^{-j2t}}{4(\alpha+j2)}\right] \\
&= -e^{-\alpha t}\left[\frac{1}{2\alpha} + \frac{\alpha\cos(2t) - 2\sin(2t)}{2(\alpha^2+4)}\right] \qquad (1)
\end{aligned}$$

1)

$$\begin{aligned}
E_x &= \lim_{T\to\infty}\int_{-T/2}^{T/2} x_1^2(t)dt = \lim_{T\to\infty}\int_0^{T/2} e^{-2t}\cos^2(t)dt \\
&= -e^{-2t}\left[\frac{1}{4} + \frac{\cos(2t) - \sin(2t)}{8}\right]\Big|_0^{T/2} \\
&= \underbrace{\lim_{T\to\infty} -e^{-T}\left[\frac{1}{4} + \frac{\cos(T) - \sin(T)}{8}\right]}_{=0} + \frac{1}{4} + \frac{1}{8} = \frac{3}{8}
\end{aligned}$$

Thus $x_1(t)$ is an energy-type signal and the energy content is 3/8.

2)

$$E_x = \lim_{T\to\infty}\int_{-T/2}^{T/2} x_1^2(t)dt = \lim_{T\to\infty}\int_0^{T/2} e^{-2t}\cos^2(t)dt + \lim_{T\to\infty}\int_{-T/2}^{0} e^{-2t}\cos^2(t)dt$$

But,

$$\begin{aligned}\lim_{T\to\infty}\int_{-T/2}^{0} e^{-2t}\cos^2(t)dt &= -e^{-2t}\left[\frac{1}{4}+\frac{\cos(2t)-\sin(2t)}{8}\right]\Big|_{-T/2}^{0}\\ &= \frac{1}{4}+\frac{1}{8}+\lim_{T\to\infty} e^{T}\left[\frac{1}{4}+\frac{\cos(T)+\sin(T)}{8}\right]=\infty\end{aligned}$$

since $2+\cos(T)+\sin(t)>0$. Thus, $E_x=\infty$ since as we have seen from the first question the second integral is bounded. Hence, the signal $x_2(t)$ is not an energy-type signal. To test if $x_2(t)$ is a power-type signal we find P_x.

$$P_x=\lim_{T\to\infty}\frac{1}{T}\int_{0}^{T/2} e^{-2t}\cos^2(t)dt+\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{0} e^{-2t}\cos^2(t)dt$$

But $\lim_{T\to\infty}\frac{1}{T}\int_0^{T/2} e^{-2t}\cos^2(t)dt$ is zero and

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{0} e^{-2t}\cos^2(t)dt &= \lim_{T\to\infty}\frac{e^T}{T}\left[\frac{1}{4}+\frac{\cos(T)+\sin(T)}{8}\right]\\ &> \lim_{T\to\infty}\frac{e^T}{T} > \lim_{T\to\infty}\frac{1+T+T^2}{T} > \lim_{T\to\infty} T=\infty\end{aligned}$$

Thus the signal $x_2(t)$ is not a power-type signal.

3)

$$E_x=\lim_{T\to\infty}\int_{-T/2}^{T/2} x_3^2(t)dt=\lim_{T\to\infty}\int_{-T/2}^{T/2}\text{sgn}^2(t)dt=\lim_{T\to\infty}\int_{-T/2}^{T/2}dt=\lim_{T\to\infty}T=\infty$$

$$P_x=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x_3^2(t)dt=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\text{sgn}^2(t)dt=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}dt=1$$

4) First note that

$$x_4^2(t)=A^2\cos^2(2\pi f_1 t)+B^2\cos^2(2\pi f_2 t)+2AB\cos(2\pi f_1 t)\cos(2\pi f_2 t)$$

Now using the identity

$$\cos^2 a=\frac{1+\cos(2a)}{2}$$

we obtain

$$\lim_{T\to\infty}\int_{-T/2}^{T/2} A^2\cos^2(2\pi f_1 t)=\lim_{T\to\infty}\frac{A^2T}{2}+\lim_{T\to\infty}\int_{-T/2}^{T/2}\frac{A^2}{2}\cos(4\pi f_1 t)=\infty$$

since $\int_{-T/2}^{T/2}\frac{A^2}{2}\cos(4\pi f_1 t)$ is bounded for all T. Using

$$\cos a\cos b=\frac{\cos(a+b)+\cos(a-b)}{2}$$

we get

$$\int_{-T/2}^{T/2}\cos(2\pi f_1 t)\cos(2\pi f_2 t)=\lim_{T\to\infty}\frac{1}{2}\int_{-T/2}^{T/2}[\cos(2\pi(f_1+f_2)t)+\cos(2\pi(f_1-f_2)t)]$$

which is also bounded for all T. So finally

$$E_x = \lim_{T\to\infty} \int_{-T/2}^{T/2} x_4^2(t) = \infty + \infty = \infty$$

Thus the signal is not of the energy-type. To test if the signal is of the power-type, note that for $f \neq 0$,

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \cos(2\pi f t) dt = 0$$

We consider two cases $f_1 = f_2$ and $f_1 \neq f_2$. In the first case

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} (A+B)^2 \cos^2(2\pi f_1 t) dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left[\frac{(A+B)^2}{2} + \frac{\cos(4\pi f_1 t)}{2} \right] dt = \frac{(A+B)^2}{2}$$

In the second case,

$$\begin{aligned} P_x &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \Big\{ \frac{A^2}{2} \left[1 + \cos(4\pi f_1 t)\right] + \frac{B^2}{2} \left[1 + \cos(4\pi f_2 t)\right] \\ &\qquad + AB \left[\cos(2\pi(f_1 + f_2)t) + \cos(2\pi(f_1 - f_2)t)\right] \Big\} dt \\ &= \frac{A^2}{2} + \frac{B^2}{2} \end{aligned}$$

Thus the signal is of the power-type, and if $f_1 = f_2$ the power content is $(A+B)^2/2$ whereas if $f_1 \neq f_2$, the power content is $(A^2 + B^2)/2$.

Problem 11. Proakis & Salehi 2.9.

Solution:
1)

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} A^2 \left| e^{j(2\pi f_0 t + \theta)} \right| dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} A^2 dt = A^2$$

Thus $x(t) = Ae^{j(2\pi f_0 t + \theta)}$ is a power-type signal and its power content is A^2.

2)

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} A^2 \cos^2(2\pi f_0 t + \theta) dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{A^2}{2} dt + \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{A^2}{2} \cos(4\pi f_0 t + 2\theta) dt$$

As $T \to \infty$, there will be no contribution by the second integral. Thus the signal is a power-type signal and its power content is $A^2/2$.

3)

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} u^2(t) dt = \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T/2} dt = \frac{1}{2}$$

The unit step signal is a power-type signal and its power content is 1/2.

4)

$$E_x = \lim_{T\to\infty} \int_{-T/2}^{T/2} x^2(t) dt = \lim_{T\to\infty} \int_{0}^{T/2} K^2 t^{-1/2} dt = \lim_{T\to\infty} 2K^2 t^{1/2} \Big|_0^{T/2} = \lim_{T\to\infty} \sqrt{2T} K^2 = \infty$$

Thus the signal is not an energy-type signal.

$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_0^{T/2} K^2 t^{-1/2} dt = \lim_{T\to\infty} \frac{1}{T}\sqrt{2T}K^2 = 0$$

Since P_x is not bounded away from zero it follows by definition that the signal is not of the power-type (recall that power-type signals should satisfy $0 < P_x < \infty$).

Problem 12. Proakis & Salehi 2.8 parts 2, 3, 4.

Solution:
2) $x_2(t+1) = \sum_{n=-\infty}^{\infty} \Lambda(t-n+1) = \sum_{n=-\infty}^{\infty} \Lambda(t-(n-1)) = \sum_{n=-\infty}^{\infty} \Lambda(t-n) = x_2(t)$. Hence $x_2(t)$ is periodic with period 1.

3) This is the sum of two periodic signals with periods 2π and 1. Since the ratio of the two periods is not rational the sum is not periodic. This can be proved as follows. The sum of two signals $x_1(t)$ and $x_2(t)$ with periods T_1 and T_2 respectively will be periodic if we can find a T such that $T = k_1 T_1 = k_2 T_2$ for some integers k_1 and k_2. Certainly, if T_1/T_2 is not a rational number, there is no such T.

4) $\sin[n]$ is not periodic. There is no integer N such that $\sin[n+N] = \sin[n]$ for all n.

Problem 13. Compute the Fourier transform of

$$f(t) = \sum_{n=-\infty}^{\infty} k_n \delta(t-n)$$

where

a. $k_n = (-1)^n$
b. $k_n = p(1-p)^n$ for $n \geq 0$ and is 0 elsewhere, $(0 < p < 1)$.

Solution: a.

$$f(t) = \sum_{n=-\infty}^{\infty} (-1)^n \delta(t-n) = \sum_{n=-\infty}^{\infty} e^{j\pi n}\delta(t-n) = e^{j\pi t}\sum_{n=-\infty}^{\infty} \delta(t-n)$$

Using the modulation property of the Fourier transform we obtain

$$F(f) = \mathcal{F}\left[\sum_{n=-\infty}^{\infty} \delta(t-n)\right]\Bigg|_{f-1/2} = \sum_{n=-\infty}^{\infty} \delta(f-n-1/2)$$

b.

$$F(f) = \sum_{n=0}^{\infty} p(1-p)^n e^{-j2\pi n f} = p\sum_{n=0}^{\infty}\left[(1-p)e^{-j2\pi f}\right]^n$$

We know that if $|r| < 1$, then $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$. In this case $|(1-p)e^{-j2\pi f}| = 1-p < 1$ and therefore

$$F(f) = \frac{p}{1-(1-p)e^{-j2\pi f}}$$

Problem 14. Compute the Hilbert transform of

a. $f_1(t) = \cos t$.
b. $f_2(t) = \sin t$.
c. $f_3(t) = \Pi(t)$.
d. $f_4(t) = \delta(t)$.
e. $f_5(t) = \frac{\sin t}{t}$.
f. $f_6(t) = \frac{1}{1+t^2}$.

Solution:

a.

$$F_1(f) = \frac{1}{2}\delta(f - \frac{1}{2\pi}) + \frac{1}{2}\delta(f + \frac{1}{2\pi})$$

$$\hat{F}_1(f) = -j\text{sign}(f)F_1(f) = -\frac{j}{2}\delta(f - \frac{1}{2\pi}) + \frac{j}{2}\delta(f + \frac{1}{2\pi})$$

$$\hat{f}_1(t) = \sin t$$

b.

$$F_2(f) = \frac{1}{j2}\delta(f - \frac{1}{2\pi}) - \frac{1}{j2}\delta(f + \frac{1}{2\pi})$$

$$\hat{F}_2(f) = -j\text{sign}(f)F_2(f) = -\frac{1}{2}\delta(f - \frac{1}{2\pi}) - \frac{1}{2}\delta(f + \frac{1}{2\pi})$$

$$\hat{f}_2(t) = -\cos t$$

c.

$$\begin{aligned} \hat{f}_3(t) &= \int_{-\infty}^{\infty} \Pi(\tau)\frac{1}{\pi(t-\tau)}d\tau = \int_{-1/2}^{1/2} \frac{1}{\pi(t-\tau)}d\tau = -\frac{1}{\pi}\ln(t-\tau)\Big|_{-1/2}^{1/2} \\ &= \begin{cases} \frac{1}{\pi}\ln\frac{t+1/2}{t-1/2} & \text{if } t < -1/2 \text{ or } t > 1/2 \\ \frac{1}{\pi}\ln\frac{-t-1/2}{t-1/2} & \text{if } -1/2 < t < 0 \\ \frac{1}{\pi}\ln\frac{t+1/2}{-t+1/2} & \text{if } 0 < t < 1/2 \end{cases} \\ &= \frac{1}{\pi}\ln\left|\frac{t+1/2}{t-1/2}\right| \end{aligned} \tag{2}$$

d.

$$\hat{f}_4(t) = \frac{1}{\pi t}$$

e.

$$f_5(t) = \frac{\sin t}{t} = \text{sinc}\left(\frac{t}{\pi}\right)$$

$$F_5(f) = \pi\Pi(\pi f)$$

$$\hat{F}_5(f) = -j\text{sign}(f)F_5(f) = -j\pi\Pi\left[2\pi(f - \frac{1}{4\pi})\right] + j\pi\Pi\left[2\pi(f + \frac{1}{4\pi})\right]$$

$$\hat{f}_5(t) = -je^{jt/2}\frac{1}{2}\text{sinc}\left(\frac{t}{2\pi}\right) + je^{-jt/2}\frac{1}{2}\text{sinc}\left(\frac{t}{2\pi}\right) = \text{sinc}\left(\frac{t}{2\pi}\right)\sin\left(\frac{t}{2}\right) = \frac{1-\cos t}{t}$$

f.

$$F_6(f) = \pi e^{-2\pi|f|}$$

$$\frac{j}{2\pi}\frac{d}{df}F_6(f) = \frac{j}{2\pi}(-2\pi)\text{sign}(f)\pi e^{-2\pi|f|} = -j\text{sign}(f)\pi e^{-2\pi|f|} = -j\text{sign}(f)F_6(f) = \hat{F}_6(f)$$

Using the property $\mathcal{F}[tx(t)] = \frac{j}{2\pi}\frac{d}{df}X(f)$, we obtain

$$\hat{f}_6(t) = \frac{t}{1+t^2}$$